Crossing Numbers and Incidence Combinatorics

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1 Euler's Formula and Crossing Numbers

In the following, let G be a graph with vertex set V and edge set E.

Recall Euler's formula from previous worksheets:

THEOREM 1 (Euler's Formula). *If G is planar, then it can be drawn in the plane. Whenever G is drawn in the plane, the drawing has f faces, where $|V| - |E| + f = 2$.*

PROBLEM 2. Show that if G is planar, then $|E| \leq 3|V|$.

Let the *crossing number* $\mathrm{cr}(G)$ of a graph G be the minimum number of pairs of edges that need to cross in order to draw G in the plane.

PROBLEM 3. To start getting bounds on crossing numbers, remember this theorem:

THEOREM 4 (Kuratowski's theorem). *A graph G is not planar if and only if some subdivision of $K_{3,3}$ or K_5 is a subgraph of G.*

1. Show that $\mathrm{cr}(G) = 0$ if and only if G is planar.
2. Show that $\mathrm{cr}(K_5) = \mathrm{cr}(K_{3,3}) = 1$.

PROBLEM 5. Show that $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$. It is actually known that $\mathrm{cr}(K_n) \leq \frac{3}{8}\binom{n}{4}$, so your lower bound is correct within a factor of 2. (Hint: Look at the contribution from each K_5-shaped subgraph.)

PROBLEM 6. Show that $\mathrm{cr}(G) \geq |E| - 3|V|$.

Now we prove the *Crossing Lemma*:

LEMMA 7 (Crossing Lemma). *If $|E| > 4|V|$, then $\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}$.*

(In fact, mathematicians have shown that $\mathrm{cr}(G) \geq \frac{|E|^3}{29|V|^2}$, but we don't care too much about the exact coefficient.)

PROBLEM 8. Let G be a graph where $|E| > 4|V|$. We will use probability to show that $\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}$.

Say you have a loaded coin, which comes up heads with probability p. (We will choose p later, but assume that $0 \leq p \leq 1$.) Now for each vertex $v \in V$, flip the coin, and put v in the set V_H if the coin comes up heads. Now let H be the graph with vertex set V_H, where $v, w \in V_H$ are connected with an edge if and only if they are in G.

1. What is the probability that a given edge e of G is in H?
2. Assume G is drawn in the plane with exactly $\mathrm{cr}(G)$ crossings, and H is drawn the same way, except with some vertices and edges missing. If e_1, e_2 are edges of G that cross, what is the probability that both are in H?
3. Recall that if X is a random variable, $\mathbb{E}[X]$ denotes the *expectation* of X, or the average value it takes. Find the expectation of these three variables:
   - $|V_H|$, the number of vertices of H
   - $|E_H|$, the number of edges of H
   - c_H, The number of crossings in the drawing of H
4. Explain why $\mathbb{E}[c_H] \geq \mathbb{E}[|E_H|] - 3\mathbb{E}[|V_H|]$.
5. Set $p = \frac{4|V|}{|E|}$. Convince yourself that this is a valid probability. Then combine the last two parts of this problem and prove the inequality

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

PROBLEM 9. Let G be a graph with n vertices each of degree at least 9. Show that the crossing number of G is at least $\frac{4n}{3}$.

2 Big O notation

We've discussed Big O notation before, but I want to use it here with multivariable functions. Let d be a natural number. Let f, g be functions from $\mathbb{N}^d$ to $\mathbb{R}$ (let's also assume they're always nonnegative). We write $f = O(g)$, and say "f is bounded above by a constant multiple of g," or "f grows at most as fast as g," when there are some $N \in \mathbb{N}, C \in \mathbb{R}$ such that for all $n_1, \ldots, n_d \geq N$, $f(n_1, \ldots, n_d) \leq Cg(n_1, \ldots, n_d)$. This notation is extremely convenient, because we don't often care about the exact constant, but is kind of ridiculous, because it uses an =-sign, when it's really more similar to $f \leq g$.

We will also write that $f = \Omega(g)$ when there are some $N \in \mathbb{N}, C \in \mathbb{R}$ such that for all $n_1, \ldots, n_d \geq N$, $f(n_1, \ldots, n_d) \geq Cg(n_1, \ldots, n_d)$, and $f = \Theta(g)$ when both $f = O(g)$ and $f = \Omega(g)$.. Note that $f = \Omega(g)$ when $g = O(f)$.

Problem 10. Let f, g be nonnegative functions from $\mathbb{N}^2$ to $\mathbb{R}$. Then show that

$$f(m,n) + g(m,n) = \Theta(\max(f(m,n), g(m,n))).$$

3 Counting Incidences with the Crossing Lemma

Let P be a set of n points in $\mathbb{R}^2$, and let L be a set of m lines. An *incidence* of P on L is defined to be an ordered pair (p, ℓ) where $p \in P, \ell \in L$, and the point p lies on the line ℓ. We will care about the number of incidences of P on L, which we denote by $I(P, L)$. The critical theorem that we will use, our hammer that makes every problem in discrete geometry look like a nail, is the *Szemerédi-Trotter* Theorem:

Theorem 11. *If P is a set of n points and L a set of m lines in $\mathbb{R}^2$, then $I(P, L) = O\left(m^{2/3}n^{2/3} + m + n\right)$.*

We will prove this in Problem 14, but first let's find some examples to explain the terms in this bound.

Problem 12. This problem will justify the m and n terms in $O\left(m^{2/3}n^{2/3} + m + n\right)$:

- For every $m, n > 0$, find a set L of m lines and a set P of n points in $\mathbb{R}^2$ such that $I(P, L) \geq m$.
- Similarly, for every $m, n > 0$, find a set L of m lines and a set P of n points in $\mathbb{R}^2$ such that $I(P, L) \geq n$.

Problem 13. This problem will justify the $m^{2/3}n^{2/3}$ term in Theorem 11:

Let $h, w \in \mathbb{N}$, and let P be the $h \times w$-grid $\{1, 2, \ldots, w\} \times \{1, 2, \ldots, h\}$, consisting of h rows of w points in the plane. Let L be the set of all lines of positive slope that pass through exactly one point of each column.

- Show that there are approximately $\frac{h}{w}$ slopes of lines in L.
- Given a slope s, (approximately) how many lines of slope s are there in L?
- Show that when h, w are both large, $|L| = \Theta(\frac{h^2}{w})$.
- Show that $I(P, L) = \Theta(|P|^{2/3}|L|^{2/3})$

We now prove Theorem 11:

Problem 14. Let P be a finite set of points and L a finite set of lines in the plane. Let the points of P divide the lines in L into line segments with endpoints in P. Define a graph G with P as its vertex set, and those line segments as its edges. Call that set of edges E.

- Can you find a condition on P and L that implies $|E| \geq I(P, L)/2$?
- Assume your condition on P and L is true. If $|E| < 4|P|$, prove that $I(P, L) = O\left(|P|^{2/3}|L|^{2/3} + |P| + |L|\right)$.
- Assume your condition on P and L is true. If $|E| \geq 4|P|$, use the crossing lemma to prove that $I(P, L) = O\left(|P|^{2/3}|L|^{2/3} + |P| + |L|\right)$.
- Making no assumptions on P and L, show that $I(P, L) = O\left(|P|^{2/3}|L|^{2/3} + |P| + |L|\right)$.

4 Distinct and Unit Distances Problems

In this section, we will investigate two questions posed by Erdős in 1946. They have stumped mathematicians for decades, but we can make some progress today!

Let P be a finite set of points in the plane.

4.1 Unit Distances

We define the *unit distance pairs* in P to be pairs $\{p, q\} \subseteq P$ such that $|pq| = 1$. We define $U(P)$ to be the number of unit pairs in P, and for a natural number n, define $U(n)$ to be the largest value of $U(P)$ for some set P with $|P| = n$. In the same 1946 paper, Erdős asked for upper and lower bounds on $U(n)$. Erdős's original upper bound has only been improved once, in 1984, and we will prove that best-known result today!

Problem 15 (Challenge). Find a sequence of sets $P_m \subset \mathbb{R}^2$ for all natural numbers m such that $|P_n| = 2^m$, and the number of unit distances in P_m is $m2^m$. Conclude that $U(n) = \Omega(n \log n)$. (This is close to the best-known lower bound.)

Problem 16. Let C be a finite set of unit-radius circles. Let $I(P, C)$ be the number of incidences of points in P on circles in C. Taking inspiration from our proof of Szemerédi-Trotter, construct a graph with vertex set P, and use it to prove that $I(P, C) = O\left(|P|^{2/3}|C|^{2/3} + |P| + |C|\right)$. Define the graph G with vertices P and the following edges: connect v and w with an edge if they are consecutive points on the same circle. Let E be the set of edges of G.

- Let T be the set of triples (p, q, c) such that $p, q \in P$ are distinct, p and q lie on $c \in C$, and p, q consecutive points on c, with q clockwise of p. Relate $|T|$ to $I(P, C)$ and to $|E|$.

- If $|E| < 4|P|$, prove that $I(P, C) = O\left(|P|^{2/3}|C|^{2/3} + |P| + |C|\right)$.
- If $|E| \geq 4|P|$, use the crossing lemma to prove that $I(P, C) = O\left(|P|^{2/3}|C|^{2/3} + |P| + |C|\right)$.

PROBLEM 17. Construct a set C of unit-radius circles such that $I(P, C)/2$ is the number of unit distance pairs in P, and use it to put an upper bound on the number of unit distance pairs.

4.2 Distinct Distances

We define the number of *distinct distances* in P to be the number of real numbers r such that there are points $p, q \in P$ such that $|pq| = r$. Erdős's *Distinct Distances Problem* asks to find upper and lower bounds on the function $d : \mathbb{N} \to \mathbb{N}$, where $d(n)$ is defined to be the minimum number of distinct distances in a set P with $|P| = n$. This problem was a famous open problem for decades, and was not fully solved until 2015.

PROBLEM 18. Use the following Theorem to put an upper bound on the distinct distances problem, by measuring the number of distinct distances in an $n \times n$ grid of points in the plane.

THEOREM 19 (Landau-Ramanujan). *The number of integers in* $[1, m]$ *that are the sum of two perfect squares is* $\theta(m/\sqrt{\log m})$.

PROBLEM 20. For each point $p \in P$, let C_p be the set of circles centered at P that pass through other points in P. Let $p, q \in P$ be distinct points. Show that $2|C_p||C_q| \geq |P| - 2$. Conclude that $d(n) = \Omega(n^{1/2})$.

Using incidences, we can prove an even better bound:

PROBLEM 21. Using our version of Szemerédi-Trotter for unit circles, show that $d(n) = \Omega(n^{2/3})$.

Hint: Let D be the set of distinct distances between points in P. Then for each $d \in D$, let C_d be the set of circles of radius d centered at points in P. Find upper and lower bounds on $I(P, \bigcup_{d \in D} C_d)$.

5 Additive Combinatorics

Let $A, B \subset \mathbb{R}$ be finite sets. Then we define $A + B$ to be $\{a + b : a \in A, b \in B\}$ and $AB = \{ab : a \in A, b \in B\}$.

PROBLEM 22. If $A \subset \mathbb{R}$ has size n, what are the minimum and maximum possible values of $|A + A|$ and $|AA|$? Given some n, can you find a set A of size n where $|A + A|$ and $|AA|$ are both maximized?

PROBLEM 23. If $A \subset \mathbb{R}$ is finite, show that $\max(|A+A|, |AA|) = \Omega(|A|^{5/4})$. (Hint: Let $P = (A+A) \times (AA) \subset \mathbb{R}^2$. Choose a set of lines L such that you can easily calculate $|I(P, L)|$, and then use Szemerédi-Trotter to find a bound on $|P|$.)

PROBLEM 24 (Challenge). Prove that $|A + AA| = \Omega(|A|^{3/2})$.

PROBLEM 25 (Challenge). Prove that $|A+A+A||AAA| = \Omega(|A|^{3/2}|A+A|^{1/2}|AA|^{1/2})$.

6 References

Most of this material comes from either Adam Sheffer's website or the notes I took in his classes. Check out `https://adamsheffer.wordpress.com/pdf-files/` if you want to learn more about this area!