Math Circle

Intermediate 2A
Functions and Proofs

18 April 2021

Glossary

- **Real numbers:** The set of real numbers is the union of the set of rational numbers and the set of irrational numbers. A real number is, therefore, any value that represents a quantity along a line.
- In mathematics, we use the following notations for the given number sets:

Number Set	Notation
Natural numbers	$\mathbb{N}$
Integers	$\mathbb{Z}$
Rational numbers	$\mathbb{Q}$
Real numbers	$\mathbb{R}$

- **Domain:** The set of all the objects a function can process as inputs is called the function's domain.
- **Target Space:** The set of all the objects the function can produce as outputs is called the function's target space.
- For a function f and sets X and Y, the notation $f : X \to Y$ denotes that the function is mapped from the set X to the set Y.
- Given $f : X \to Y$, the **image** of x in X is $f(x)$. Images are elements of the range. The preimage of y is the set of all x in X such that $f(x) = y$. Preimages are subsets of the domain.
- **Cardinality:** Cardinality measures the size of a set. For a finite set, it equals the number of elements in the set. For infinite sets, cardinality allows us to compare the sizes of infinite sets and talk about different types of infinity.
- $\exists$ "there exists"—this is formal math notation.
- $\forall$ "for all"—this is formal math notation.

Warm Up Problems

1. Make a Venn diagram showing the relations between the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

2. Is $p(x) = \pm\sqrt{x}$ a function? Why or why not?

3. What are the domains and ranges of the following functions?

   (a)

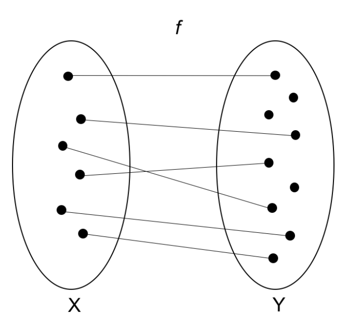
Mathematician	Place of Birth
Maria Agnesi	Milan, Italy
Sophie Germain	Paris, France
Ada Lovelace	London, England
Sofia Kovalevskaya	Moscow, Russia
Emmy Noether	Erlangen, Germany

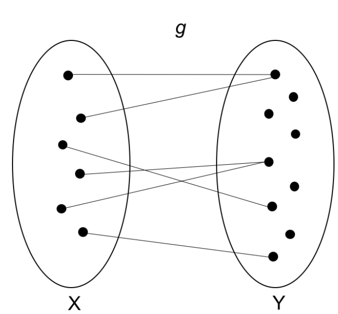
(b) $f(x) = x$

(c) $f(x) = \sqrt{x} + 1$

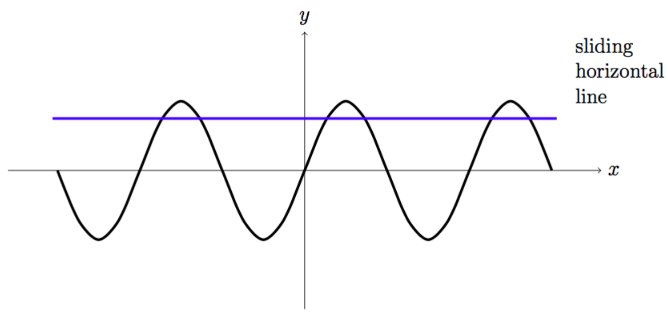
One-to-One Functions

A function is called **one-to-one** if every element of its range corresponds to exactly one element of the domain. One-to-one is often written as 1-1. For example, function f below is one-to-one while function g is not.



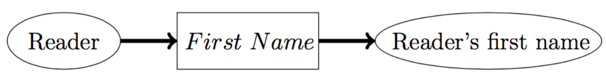


There exists a geometric way, called the horizontal line test, to determine whether a function is one-to-one. Take a ruler, place it horizontally on the graph, and slide up and down, keeping the edge horizontal. If the edge ever intersects the graph at more than one point, then the function is not one-to-one. Otherwise, it is. For example, the test shows that the following function is not one-to-one.

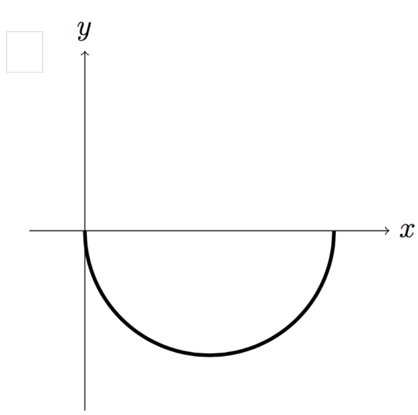


1. Are the following functions one-to-one? Why or why not?

    (a) "Reader" below refers to the readers of a specific book.



    (b)



(c) $f(x) = 2x$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

   (a) Is f one-to-one? Why or why not?

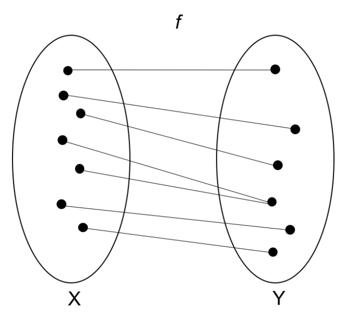
   (b) Is it possible to find a subset $X \subset \mathbb{R}$ such that $f : X \to \mathbb{R}$ is one-to-one?

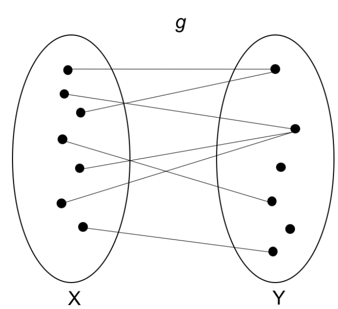
   (c) Find three different sets X such that $f : X \to \mathbb{R}$ is one-to-one.

3. Give your own example of a function that is one-to-one and a function that is not one-to-one.

Onto Functions

A function f from a set A to a set B is called **onto** if for every element b in B (the target set), there exists an element in A (a preimage) such that $f(a) = b$. For example, function f below is onto while function g is not.





1. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = 2x$.

   (a) Is $f(x)$ onto?

   (b) What are the target set and range of this function?

(c) Can you find a set X such that $f : \mathbb{Z} \to X$ is onto?

2. Let $f : \mathbb{Q} \to \mathbb{Q}$ be given by $f(x) = 2x$. Is f onto? Explain why or why not.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$. Is f onto? Explain why or why not.

4. Give your own example of a function that is onto and a function that is not onto.

Bijective Functions

A function that is one-to-one and onto is called a **bijection**. A bijective function establishes an equivalence of its domain and its target set as sets, meaning that each and every element of the target set has one, and only one, preimage in the function's domain. For example, counting is nothing but establishing a bijection between the elements of some finite set and the first n elements of the set N of natural numbers, $1, 2, 3, ...n$.

1. Give your own example of a function that is both one-to-one and onto.

2. Let A and B be finite sets of cardinalities $|A|$ and $|B|$, respectively. Fill in the blanks with $\geq, =$ or $\leq$.

   (a) If $\exists$ a one-to-one function $g : A \rightarrow B$, then $|A|$ ____ $|B|$.

   (b) If $\exists$ an onto function $g : A \rightarrow B$, then $|A|$ ____ $|B|$.

   (c) If $\exists$ a bijective function $g : A \rightarrow B$, then $|A|$ ____ $|B|$.

Proofs[1]

1. Show that all linear functions of the form $f(x) = ax + b$, where a, b are real numbers and $a \neq 0$, are one-to-one functions.

2. (*)Prove that $x + \frac{1}{x} \geq 2$, if $x > 0$.

[1]Problems marked with an asterisk (*) in this section have been adapted from the book "A Moscow Math Circle" by Sergey Dorichenko.

3. (*)Consider a plane on which any point is colored red or green. Prove that one can find two points exactly 1 inch apart on this plane that are the same color.

4. (*)An airport has fifteen gates and various moving walkways, each of which connects exactly two gates. From each gate, you can reach at least seven others via the walkways. Prove that it is possible to go from any gate to any other via either one or two walkways.

5. (*)Terry walks on a trail and meets a group of five people who walk in the opposite direction. Prove that in this group, either at least three people know Terry, or at least three people don't know her.

   (a) **(*)Challenge:** Prove that among any six people, it is always possible to find three that know one another, or else there are three mutual strangers.

6. **Challenge:** Show that all functions of the form $f(x) = a \cdot (x - h)^2 + k$, where $x \geq h$, a, h, k are real numbers and $a \neq 0$, are one-to-one functions.