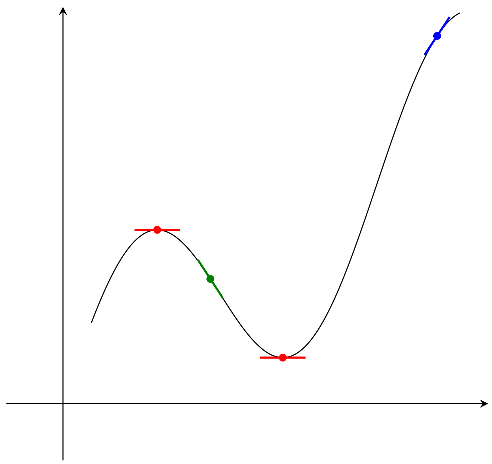
ORMC: BASIC LINEAR ORDINARY DIFFERENTIAL EQUATIONS

OLYMPIAD GROUP 1, WEEK 2

Derivatives. The derivative of a function f at a point x is

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

and intuitively represents the *slope of the tangent*.



Intuition. Derivatives measure change:

$$\boxed{f'(x) = \text{How quickly } f \text{ increases with } x \text{ (if } f' < 0 \text{ then } f \text{ decreases).}}$$

$$\mathit{Velocity}(\mathit{Time}) = \mathit{Position}'(\mathit{Time}) = \text{How quickly } \mathit{Position} \text{ increases with } \mathit{Time}.$$

$$\mathit{Acceleration}(\mathit{Time}) = \mathit{Position}''(\mathit{Time}) = \text{How quickly } \mathit{Velocity} \text{ increases with } \mathit{Time}.$$

Of course, $f'(x)$ may not be well-defined in general (f might not be *differentiable* at x). We won't worry about that in these lectures; we'll only work with differentiable functions. It is not hard to deduce the following formulas directly from the definition of the derivative (where c is a constant):

$$(f \pm g)' = f' \pm g', \qquad (c \cdot f)' = c \cdot f', \qquad (f \cdot g)' = f' \cdot g + f \cdot g',$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}, \qquad (f \circ g)' = (f' \circ g) \cdot g'.$$

Also, a constant function $f(x) = c$ has $f'(x) = 0$ everywhere (so abusing notation a bit, $c' = 0$ when c is a constant). The converse is also true (although we won't prove it):

Fact. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. If $f'(x) = 0$ for all x, then f is a constant function (i.e., there is a constant c such that $f(x) = c$ for all x). In other words, the only solutions of the differential equation

$$f' = 0$$

are the constant functions. The same is true if we restrict our function f to an interval.

Problem 1. Suppose that f, g are such that $f' = g'$ everywhere. Show that $f - g$ is a constant function.

Problem 2. (Linear functions) Let a, b be constants.

(a) Show that $x' = 1$. Use this to show that $(ax + b)' = a$.

(b) Find all functions f satisfying the differential equation

$$f' = a.$$

Problem 3. (Polynomials) Let n be a positive integer.

(a) Show that $(x^n)' = nx^{n-1}$.

(b) Let $a_0, a_1, \ldots, a_n$ be constants. Compute the derivative

$$\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)'.$$

(c) Find all functions f satisfying the differential equation

$$f'(x) = 3x^2 + 4x + 5.$$

Problem 4. (Gravity) We throw a ball upwards and denote its position at time t by $y(t)$ (initially we have $y(0) = 0$; the greater the value of y, the higher our ball is). Due to the gravitational force, which induces a constant acceleration downwards, we know that (while our ball is in the air)

$$y''(t) = -g,$$

where g is a constant. We also know the speed v_0 with which our ball was thrown upwards, meaning that

$$y'(0) = v_0.$$

Determine $y(t)$ in terms of v_0 and g, for all t until our ball hits the ground again. At what time t will this happen?

Problem 5. (Exponentials)

(a) Suppose that a function f satisfies the differential equation

$$f' = f.$$

Show that f cannot be a polynomial.

(b) But what if we used infinitely many powers of x? Explain, informally, why the summation

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$$

should satisfy $\exp' = \exp$. If we denote $e := \exp(1)$, then we have, in fact, $\exp(x) = e^x$.

(c) You can assume that $f(x) = e^x$ is a solution of $f' = f$. Show that the only other solutions have the form $f(x) = ce^x$ for some constant c. *Hint: Consider the derivative of* $e^{-x} f(x)$.

Problem 6. (Damped movement) We give a good push to a cube on a long horizontal table. Denote by $v(t)$ the cube's speed at time t, and let $v_0 = v(0)$. Assume that the friction force with the table (which slows down our cube) is proportional to the cube's speed, so we have

$$v'(t) = -\alpha v(t),$$

for some constant $\alpha > 0$. Show that the speed of the cube at any given time $t \geqslant 0$ is

$$v(t) = v_0 e^{-\alpha t}.$$

Does the cube ever stop moving in this model?

Problem 7. (Oscillations)

(a) Explain, informally, why the summations

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$$

should satisfy $\sin' = \cos$ and $\cos' = -\sin$.

(b) Many physics problems (such as harmonic oscillators) lead to the differential equation

$$f''(t) = -\omega^2 f(t),$$

for some $\omega > 0$. Show that the functions

$$\sin(\omega t) \qquad \text{and} \qquad \cos(\omega t)$$

satisfy this differential equation. Can you find any other solutions? *Hint: Linear combinations.*

Below is a summary of useful formulas (where c is a constant):

$(f+g)' = f' + g'$	$c' = 0$
$(c \cdot f)' = c \cdot f'$	$x' = 1$
$(f \cdot g) = f' \cdot g + f \cdot g'$	$(x^n)' = nx^{n-1}$
$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$	$(e^{cx})' = ce^{cx}$
$(f \circ g)' = (f' \circ g) \cdot g'$	$\sin' = \cos, \quad \cos' = -\sin$

Homework

Problem 1. (Integrating factor method) Let g be a given differentiable function, and consider the differential equation

$$f' = g' \cdot f.$$

(a) Show that $f(t) = c \cdot e^{g(t)}$ is a solution to the equation above (c is any constant).

(b*) Show that these are *all* solutions, by computing the derivative of $e^{-g(t)} f(t)$. (This is similar to Problem 5c).

(c) Use the method above to find the solutions to $f'(t) = t \cdot f(t)$.

Problem 2. (Damped falling object) Consider the example in Problem 4, but in which one also takes *air drag* into account. That is, besides the gravitational acceleration g, we also have an acceleration as in Problem 6. Combined, they give the differential equation

$$y''(t) = -g - \alpha y'(t).$$

Plug in the guess $y(t) = e^{rt}$ into the equation above, and find out for which r it works. You may assume that any square-roots that you encounter are well-defined.