Random Events II

Los Angeles Math Circle

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Random Events

1. We remind ourselves of the definitions and ideas from last week with this Monopoly board problem.

   The Monopoly board consists of 40 spaces. Players start at Go (space 0) and work their way clockwise around the board. For these questions, we will consider the "Go to Jail" space to be blank, and use only a single die, instead of the typical two used for the game.

   (a) How many spaces would you expect to move in a turn?

   (b) How many turns would you expect to take to pass Go again?

(c) Roughly calculate the probability of landing on Boardwalk (space 39), given that you're starting at Go (space 0), using one die and continuously rolling.

i. What is the probability of landing on space 1?

ii. What is the probability of landing on space 2?

iii. Is there an easy way to calculate the probability of landing on the tenth space? Why or why not?

iv. Let's go back to the meaning of expectation. Since the expectation of a dice roll is $\frac{7}{2}$, it implies that in the long-term, if we keep rolling the die, for every 7 spaces we move, we should land on _______ spaces.

v. So, what is the probability of landing on the n^{th} space (assuming that n is somewhat large)?

vi. What is the probability of landing on Boardwalk?

Expectation of Two Random Events

In some cases, we are going to consider two random events. For example, not only is the outcome of a dice roll a random event, but the number of rolls is also a random event. That is,

$$E[S] = E[X + X + \cdots + X], \quad \{X \text{ summed } N \text{ times}\}$$

Where N is a random whole number itself representing the number of dice rolls, and X is the random variable representing the outcome of a dice roll.

If the two rnadom variables, X and N, are independent of each other, it is easy to find the expectation.

1. Express the expectation of S as the expectations of the two independent random variables, X and N.

   (a) Express the expectation of S in the form of random variables X and N.

   (b) What is $E[cX]$ equivalent to, if c is a constant? (Hint: recall the property of taking the expectation of multiplying a random varaible by a constant.)

   (c) Is the value of c independent of a random event, X?

   (d) Using the fact that X and N are independent random variables, what would you intuitively assume is the expression for $E[S]$?

   (e) What would happen if the random variables N and X were not independent of each other?

2. **Expected Value of a Dice Game**
Sam is deciding whether or not to play a dice game that costs \$12.50 to play. The game is played as follows: Same will first roll a red die that will give him an outcome N. He will thenroll N green dice and he will be paid for the sum of all the values that appear on the green dice.

   (a) Should Same expect to make money from the game? (Asuume all dice are fair and six-sided.)

   (b) What if the game costs \$12?

Probability (So Far)

1. Suppose a random event has only two outcomes, A and B. We can express the probability of each outcome as $P(A)$ and $P(B)$.

   (a) What does the sum $P(A)$ and $P(B)$ equal?

   (b) In probability theory, we say that the complement of an outcome, A, is the union of all other outcomes. The complement of A is expressed as A^C. What is $P(A^C)$ equal to in terms of $P(A)$?

   (c) What is A^C in this example?

2. If Ivy flips three fair coins, what is the probability she does not get three tails?

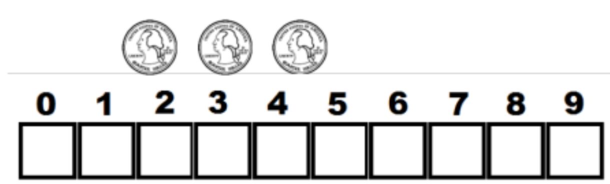
   (a) What is the probability Ivy flips three tails?

   (b) Now use part (b) in problem 1 to solve the question.

   (c) How would you solve this problem without using complements?

3. You are on a game show where you receive \$10,000 if you win and nothing if you lose. The game is played on a simple board: a track with sequential spaces numbered from 0 to 1,000. The zero space is marked "start," and your token is placed on it. You are handed a fair six-sided die and **one** coin. You are allowed to place the coin on a nonzero space. Once placed, the coin may not be moved.

   After placing the coin, you roll the die and move your token forward the appropriate number of spaces. If, after moving the token, it lands on the space with the coin on it, you win. If not, you roll again and continue moving forward. If your token passes the coin without landing on it, you lose. On which space should you place the coin to maximize your chances of winning? (We will answer this on the following pages.)



Early Placement Probabilities

(a) Why would we prefer to put our coin on a space after 3 and not before?

(b) Suppose we put the coin on space 5. What is the probability we land on it on our first roll?

(c) What is the probability we do not land on it on our first roll and land on it on our second roll?

(d) What is the probability we land on space 5 on our first or second roll?

(e) Now suppose we put the coin on sapce 6. What is the probability we land on it on our first roll?

(f) What is the probability we do not land on it on our first roll and land on it on our second roll?

(g) What is the probability we land on space 5 on our first or second roll? Between spaces 5 and 6, which is the better space for placing the coin?

(h) What is the probability we land on space 6 on our first or second roll? (solve without using combinations)

(i) Comparing the three coin placement, which do you think is the best placement? Have we calculated the exact probabilities of landing on these spaces?

(j) Why are we able to make these estiamtes with only the first two rolls? (hint: what is the probability that it takes 6 rolls to land on the $6.^{th}$ space?)

Later Placement Probabilities

(a) Now we will use our findings from the Monopoly problem. What is the probability that we land on any n^{th} space, assuming n is somewhat large ($n > 10$).

(b) What is the probability of landing on the 998^{th} space? Is the probability different than landing on the 1000^{th} space?

(c) Now compare your answer in part (b) with the probability of landing on space 6. Is there a shortcut for calculating which fraction is larger?

(d) What can you conclude is the best placement for the coin?

(e) What would you guess are the best spaces for two coins? In this case, you are handed two coins that you must place on two non-zero spaces. If, after moving the token, it lands on a space with a coin on it, you win.

Binomial Coefficients

Let us take a look at algebraic identities.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$

In order to understand the form of the equality above, let us expand its left-hand side and write out all the factors in all the summands.

$$(a+b)^2 = (a+b)\cdot(a+b) = aa+ab+ba+bb$$

Similarly,

$$(a+b)^3 = aaa+aab+aba+abb+baa+bab+bba+bbb$$

Notice that the expansion of $(a+b)^2$ consists of all 2-letter arrangment with repetitions of the letter a and b. And the expansion $(a+b)^3$ consists of all 3-lettter arrangements with repetitions of the letters a and b.

1. More generally, how many letters will be in each term in the expansion for $(a+b)^n$?
2. If a appears k times in one of the terms, how many times will b appear in that term?

3. Write down a formula to calculate the number of terms which have k letters a and $n-k$ letters b.

4. After simplifying the expansion by collecting terms, what would be the coefficient of the following?

   (a) $a^n b^0$

   (b) $a^{n-1} b^1$

   (c) $a^{n-2} b^2$

   (d) $a^{n-3} b^3$

   (e) $a^{n-k} b^k$

   (f) $a^0 b^n$

5. Fill in the blanks with the coefficients you found above.

$$(a+b)^n = \_\_\_\_\_\_a^n b^0 + \_\_\_\_\_\_a^{n-1} b + \_\_\_\_\_\_a^{n-2} b^2 + \_\_\_\_\_\_a^{n-3} b^3 + \cdots + \\ + \_\_\_\_\_\_a^{n-k} b^k + \cdots + \_\_\_\_\_\_a^0 b^n$$

This formula is called the **binomial expansion**. Since $a+b$ is a polynomial made of two terms, it is called the **binomial**. The coefficients of the variables in the expansions given above are called **binomial coefficients**.

Binomial theorem for any positive integer n:

$$\textbf{Binomial Theorem} : (a+b)^n = \sum_{k=1}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k.$$

Where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. Take a minute or two to fully comprehend this formula and see if the summation notation makes sense to you. Ask your instructor if you have any questions!

6. We can now arrange the coefficients in these expansions as show below. Fill in the next two rows. This array of numbers is call **Pascal's Triangle**, after the name of the French mathematician Blaise Pascal. Can you spot another pattern?

Power of *(a+b)*	**Binomial Coefficients**
1	1
2	1 2 1
3	1 3 3 1
4	
5	

Binomial Probabilities

Consider Flipping a coin with probability of getting heads equal to p and probability of getting tails equal to $1-p$ (if $p=\frac{1}{2}$, it is the usual fair coin).

The probability getting k head if you flip the coin n times is:

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

1. What do p and $(1-p)$ sum to?

2. Let $a=p$ and $b=1-p$. Write down the statement of the binomial theorem for this case.

3. What does the right side of this formula represent? Why is this statement intuitively true?

4. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.