ORMC: LINEAR RECURRENCES

OLYMPIAD GROUP 1, WEEK 1

Definition. A linear recurrence of degree k is an equation of the form

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k} \tag{1}$$

where $a_1, \ldots, a_k$ are fixed constants. One frequently wishes to find sequences x_n that satisfy this, starting from some initial known values $x_1, \ldots, x_k$.

[*Examples: powers of 2, Fibonacci*] [*Uniqueness given initial conditions*]

Problem 1. (Textbook example)

(a) Show that if r satisfies

$$r^k = a_1 r^{k-1} + a_2 r^{k-2} + \cdots + a_k,$$

then the sequence $x_n := r^n$ is a solution of the recurrence (1).

(b) Show that the Fibonacci sequence ($F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$) is given by the formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Problem 2.

(a) Find the general term x_n given by the recurrence

$$x_1 = 1,\ x_2 = 1, \quad x_n = 2x_{n-1} - x_{n-2}.$$

(b) Find the general term x_n given by the recurrence

$$x_1 = 1,\ x_2 = 2, \quad x_n = 2x_{n-1} - x_{n-2}.$$

Compute a few small values, and conjecture an answer. Then prove it by induction. *Is this of the form given in Problem 1?*

(c) Now let a and b be any fixed numbers. Find the general term x_n given by the recurrence

$$x_1 = a,\ x_2 = b, \quad x_n = 2x_{n-1} - x_{n-2},$$

using parts (a) and (b).

[*Main idea: check solutions of the form $x_n = n^d \cdot r^n$.*]

Theorem. (Hard) Any linear recurrence of degree k can be solved by considering linear combinations of particular solutions of the form $n^d \cdot r^n$, where r is a root of $t^k - a_1 t^{k-1} - \cdots - a_k$. Moreover, one needs only look at d less than the *multiplicity* of r in this polynomial.

Problem 3. *Fact:* For any $d \geqslant 1$, there is a linear recurrence of degree $d+1$ such that $x_n = n^d$ is a solution of it. Notice that for $d = 1$, we have already seen such a recurrence in Problem 2.

Now find such a recurrence for the case $d = 2$. In other words, find a_1, a_2, a_3 such that for all n,

$$n^2 = a_1(n-1)^2 + a_2(n-2)^2 + a_3(n-3)^2.$$

Problem 4. Let $p > 5$ be a prime number such that 5 is a perfect square modulo p (i.e. there exists an integer z such that $z^2 \equiv 5 \pmod{p}$.) Show that F_{p-1} is divisible by p. *Be careful about how you use problem 1(b)!*

Homework

Problem 1. (Distinct roots) Solve the linear recurrence

$$x_0 = 2,\ x_1 = 5, \quad x_n = 5x_{n-1} - 6x_{n-2}$$

Problem 2. (Double root) Solve the linear recurrence

$$x_0 = 1,\ x_1 = 4, \quad x_n = 4x_{n-1} - 4x_{n-2}.$$

The formulas you get should be much simpler than for the Fibonacci sequence.