Lesson 1: Combinations

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For a positive integer n and $0 \leq k \leq n$ let $\binom{n}{k}$ denote the number of ways to choose k objects out of n distinct objects. For example, $\binom{3}{1} = 3$ and $\binom{4}{2} = 6$.

Problem 1.
Show that

$$\binom{n}{k} = \binom{n}{n-k}$$

without using the formula in problem 2.

Problem 2.
Show that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Problem 3.
If $n > 1$ and $k > 0$, show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

a) Algebraically, using the formula from problem 2.

b) Combinatorially, without using the formula.

Problem 4.
Consider an $n \times m$ square grid. Show that the number of ways to travel along the grid lines from the bottom-left corner to the upper-right corner while always going up or right is equal to $\binom{n+m}{n}$.

Problem 5.
a) Show that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

b) Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0$$

Problem 6.
Let $\triangle ABC$ be a right triangle with $\angle B = 90°$. If BH is the altitude, $AH = a$ and $HC = b$, show that $BH = \sqrt{ab}$.