GRAPH COLORINGS

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OLGA RADKO MATH CIRCLE ADVANCED 3

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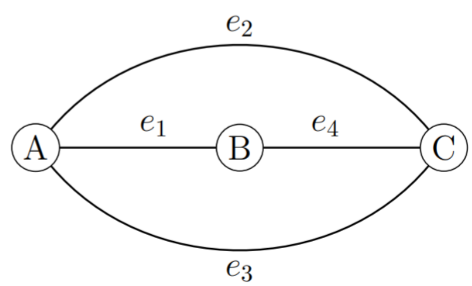
1 Optional: Review of planar graphs

Recall that a graph G is a pair (V, E), where V is a set called the set of *vertices* and E is a set consisting of pairs of elements in V, called *edges*. We can draw graphs by drawing vertices as dots or circles, and edges by lines between them. Note that the same graph can be drawn in many different ways. A graph is called *planar* if it can drawn in the plane in such a way that no edges cross one another.

Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior face* or *infinite face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1 bounded by the edges e_1, e_2, and e_4, and F_2 bounded by the edges e_1, e_3, and e_4. Its exterior face F_3 is bounded by the edges e_2 and e_3.

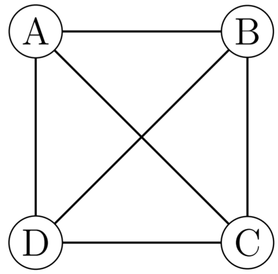


The *Euler characteristic* of a graph is the number of the graph's vertices minus the edges plus the number of the faces,

$$\chi = V - E + F.$$

Problem 1. Show that each of the following graphs are planar by drawing them without intersecting edges, then compute the Euler characteristic of the graphs. Can you make a conjecture about the Euler characteristic of every planar graph?

1.



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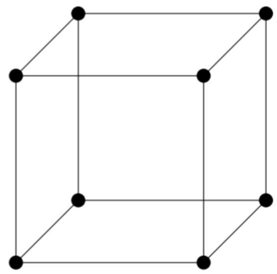


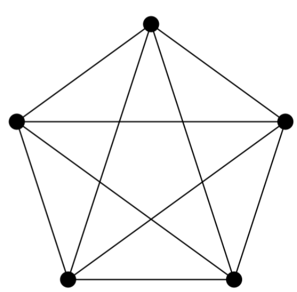
Theorem 1 (Euler's characteristic formula)**.** Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.

Remember that connected just means that you can go from any vertex to any other vertex by travelling along edges. All of the graphs in this worksheet are connected.

Problem 2. We will prove Euler's characteristic formula in this problem.

1. A tree is a graph with no cycles (i.e. a way to go out of a vertex and come back without reusing edges). Explain why a tree is planar. Explain why the Euler characteristic of a tree is 2.
2. Prove that the Euler characteristic of a planar graph is always 2.

We want to develop a method to show that certain graphs are *not* planar. Let's study a motivating example. This graph is called K_5.



Problem 3.

1. Let a finite connected simple graph planar graph have $E > 1$ edges and F faces. Prove that then $2E \geq 3F$.
2. Prove that then $E \leq 3V - 6$.
3. Explain why K_5 is not planar.

Amazingly, there are two graphs that are in some sense, "representative" of all non-planar graphs! K_5 is one of these graphs.

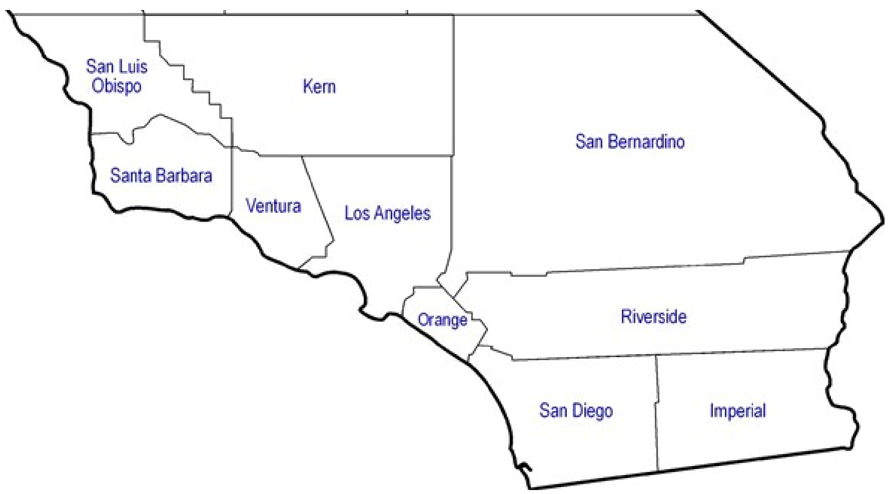
Theorem 2 (Kuratowski's theorem)**.** A graph G is not planar if and only if some subdivision of $K_{3,3}$ or K_5 is a subgraph of G.

2 Introduction to graph coloring

Today's topics revolve around coloring graphs and maps.

Problem 4.

1. Using as few colors as possible, color the counties of southern California so that no two counties sharing a land border have the same color (corners that touch are not necessarily borders). Argue why using fewer colors is not possible.



2. Recall that a graph is a set of vertices connected by edges. How can we model a map by a graph in way that allows us to convert map coloring questions to a question about the graph?

Note that not only can maps be represented by graphs, they are represented by *planar graphs*! This means we'll have all of the tools from the theory of planar graphs at our disposal when we study maps. We'll explore the minimum number of colors needed to color maps, using the correspondence between maps and planar graphs.

To build up the theory of colorings, we will first discuss some generic examples coloring vertices of (possibly non-planar) graphs. A k-coloring of a graph G is a labelling of the graph's vertices with k colors such that no two vertices sharing the same edge have the same color. The smallest k such that a k-coloring exists is called the *chromatic number* of the graph, and denoted $\chi(G)$. The number of k-colorings of G is denoted $\chi_G(k)$, and it turns out that $\chi_G(k)$ is always a polynomial! We will prove this in the next section.

Problem 5. Compute the chromatic numbers of the following graphs.

1. The path graph P_n has n vertices numbered $1, \ldots, n$ and $n-1$ edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}$.
2. The complete graph K_n has n vertices and an edge between any two distinct vertices.
3. A connected tree T on n vertices.
4. The cycle graph C_n on n vertices numbered $1, \ldots, n$ and exactly n edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$.

3 Chromatic polynomials

Before proving the main theorem of this section, which is that $\chi_G(k)$ is a polynomial, let's explore some examples.

Problem 6. The path graph P_n has n vertices numbered $1, \ldots, n$ and $n-1$ edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}$. Compute the chromatic polynomial of P_n.

Problem 7. The complete graph K_n has n vertices and an edge between any two distinct vertices. Compute the chromatic polynomial of K_n.

Theorem 3 (Deletion-Contraction Recurrence)**.** Let G be a graph and e be an edge of G. Denote by $G - e$ the graph obtained from G by deleting e (and leaving its two endpoints untouched). Denote by G/e the graph obtained from G by contracting e, i.e. deleting e, then merging its two endpoints into one. Then

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k).$$

Problem 8. Prove the Deletion-Contraction Recurrence.

Theorem 4. Let G be a graph and let $\chi_G : \mathbb{N} \to \mathbb{N}$ be the function where $\chi_G(k)$ is the number of distinct k-colorings of G. Then $\chi_G(k)$ is a polynomial.

Problem 9. Prove, by induction, that $\chi_G(k)$ is a polynomial.

Problem 10. 1. Show that the degree of $\chi_G(k)$ is the number of vertices of G.

2. Show that $\chi_G(k)$ has leading coefficient 1.

3. Let G be a vertex on n vertices and m edges. Show that the coefficient of the k^{n-1} term is $-m$.

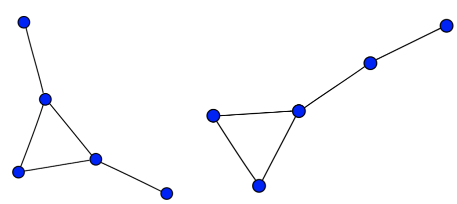
Problem 11. Find the chromatic number and polynomial of the following graphs (and explain your reasoning).

1. Any tree T on n vertices. (Surprisingly, all trees on n vertices have the same chromatic polynomial! Recall that a tree is a graph that has no cycles, and that a tree with n vertices always has $n-1$ edges.)

2. The cycle graph C_n on n vertices numbered $1, \ldots, n$ and exactly n edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$.

Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. For instance, Problem 11 part 1 shows that all trees are chromatically equivalent.

Problem 12. Show that the following graphs are chromatically equivalent.



An *isomorphism of graphs* G and H is a bijection $f : V(G) \to V(H)$ between the vertex sets of G and H such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. A graph is *chromatically unique* if it is determined by its chromatic polynomial up to graph isomorphism.

Problem 13. 1. Prove that the graphs in Problem 12 are non-isomorphic. This is a great example of chromatically equivalent graphs that are not isomorphic.

2. Find another non-isomorphic graph with the same chromatic polynomial as the two in Problem 12.

Problem 14. For $n \geq 3$, show that the cycle graph C_n is chromatically unique.

4 The five color theorem

We now return to the problem of coloring maps and planar graphs.

Problem 15.

1. Draw a map (or equivalently, a planar graph) that requires at least 4 colors.
2. Describe a graph that requires at least 100 colors. Is your graph planar? Can you draw a planar such graph?

As alluded to by the title of this section, we will prove the following theorem.

Theorem 5 (Five color theorem)**.** If G is a planar graph, then $\chi(G) \leq 5$.

Recall that the degree of a vertex is the number of edges coming out of it. Here is one fact that proof requires.

Problem 16. Show that every planar graph has a vertex of degree at most 5. (Hint: You might want to use a result from the planar graphs section.)

We can now prove the five color theorem!

Problem 17. We will use induction on the number of vertices.

1. Prove the base case.
2. Now assume that all planar graphs on $n-1$ vertices can be colored with 5 colors, and we want to show that all planar graphs on n vertices can too. Let v be a vertex of degree at most 5.
   (a) If v actually has degree at most 4, prove the claim.
   (b) Now suppose v has degree exactly 5. Why can we assume that the neighbors of v use all 5 colors?
   (c) Suppose the 5 neighbors of v are colored (in clockwise order) red, yellow, green, blue, and purple. If there is no red-green alternating path between the red vertex and the green vertex, prove the claim.
   (d) Suppose there is a red-green alternating path between the red vertex and the green vertex. Show that there is no yellow-blue alternating path between the yellow vertex and the blue vertex, and then finish the proof of the theorem.

We proved the five color theorem, but it turns out that four colors always suffice for planar graphs! In 1976, Kenneth Appel and Wolfgang Haken proved the *four color theorem*, which states that any planar graph can be colored with just 4 colors! Even though mathematicians have made progress in simplifying their original proof, which involved 1834 different cases to be checked by a computer, today's best known proof still involves 1492 cases to check. (You can see a modern computer proof here: https://github.com/math-comp/fourcolor)

5 Ramsey theory

So far, we have been interested in coloring vertices, but what about edges? There are many ways that edge colorings can be defined, and in this section we will look at red-blue colorings, i.e. every edge is colored either red or blue. (There are also notions of edge coloring that map more closely to our vertex coloring, such as requiring that edges that meet the same vertex have different colors, but that's for another time.)

The number $R(r, s)$ called the Ramsey number for red r and blue s, and is the smallest n such that every red-blue edge coloring of K_n contains either a red K_r as a subgraph, or blue K_s as a subgraph. Recall that K_n is the complete graph on n vertices, meaning every pair of vertices is connected by an edge. One popular story to make this definition clearer is that $R(r, s)$ is the smallest n such that out of n people, we are guaranteed at least r to be mutual friends or s to be mutual strangers.

Problem 18. Find $R(2, n)$, for all $n \geq 1$. That is, what is the smallest group of people in which we are guaranteed at least 2 mutual friends or n mutual strangers?

Problem 19. Show that $R(3, 3) = 6$.

Problem 20. In this problem, we will show that $R(3, 4) = 9$.

1. First prove that $R(3, 4) > 8$.
2. To show $R(3, 4) \leq 9$, pick a vertex v. Consider the cases where v is incident with at least 4 red edges.
3. Consider the case where v is incident with at least 6 blue edges.
4. In order to study the final case, we will prove the handshaking lemma. Show that the number of vertices of odd degree is always even.
5. As our final case, draw a contradiction when v is incident with fewer than 4 red and fewer than 6 blue edges.

One fact that is not immediately clear is whether or not $R(r, s)$ even exists for all natural numbers r and s. In other words, is it necessarily true that you are guaranteed monochromatic complete subgraphs of any size, as long as you take the original graph to be much larger? In 1928, Frank Ramsey proved that you are indeed guaranteed this, and we will show it now too.

Theorem 6 (Ramsey's theorem)**.** $R(r, s)$ is finite for all natural numbers r, s.

Problem 21. We will prove the theorem in the following steps:

1. Show $R(r, 1)$ and $R(1, s)$ exist for all natural numbers r and s.

2. Prove that $R(r, s) \leq R(r-1, s) + R(r, s-1)$.
3. Deduce Ramsey's theorem.

Computing $R(r, s)$ is difficult for even moderately large r and s. As of today, mathematicians do not even know the value of $R(5, 5)$.

Problem 22. Find the best upper and lower bounds on $R(5, 5)$ that you can.

The late mathematician Paul Erdős, who was one of the greatest mathematicians of the 20th century, once said the following: "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

More than just calculating values of $R(r, s)$, Ramsey theory is an active field of mathematics research where we try to find order amidst chaos. For instance, no matter how chaotic the red-blue coloring of K_6 may be, we can still find order because it must contain a monochromatic K_3. In general, this field is about asking: how large should objects be in order to guarantee the existence of a subobject satisfying some property?

6 Hadwiger-Nelson Problem

In this section, we will examine an open problem that pertains to a particular graph U. The vertex set of the graph is the plane, $\mathbb{R}^2$, and two points $(x_1, y_1), (x_2, y_2)$ are connected by an edge if and only if the the distance between them is 1 (that is, $(x_1-x_2)^2+(y_1-y_2)^2 = 1$). This is sometimes known as the *unit distance graph*. The problem is to determine the chromatic number $\chi(G)$.

Problem 23. Prove that $3 \leq \chi(U)$ by finding a subgraph of U that is not 2-colorable.

Problem 24. Find a 9-coloring of U by tiling the plane with a grid of squares and coloring the squares.

Problem 25. Now we will work to find a subgraph of G that is not 3-colorable, thus showing that U is not 3-colorable.

1. Find a subgraph of U with 4 vertices a, b, c, d such that in any 3-coloring, a and d have to have the same color.
2. Using a few copies of that subgraph, build a subgraph of U with 7 vertices and 11 edges that is not 3-colorable.

Problem 26. Find a 7-coloring of U by tiling the plane with hexagons.