Combinatorics Homework

1. Find all possible values of k such that

$$\frac{(2k)!}{(2k-1)!} = \frac{(k+6)!}{(k+5)!}$$

$$\frac{(2k)\cdot\cancel{(2k-1)}\cdots}{\cancel{(2k-1)}\cancel{(2k-2)}\cdots} \to 2k$$

$$\frac{(k+6)\cdot\cancel{(k+5)}\cancel{(k+4)}\cdots}{\cancel{(k+5)}\cancel{(k+4)}\cdots} \to k+6$$

$$2k = k+6 \to 2k-k=6$$

$$\boxed{k=6}$$

2. Simplify $\frac{(x^2-4)!}{(x-2)\,(x^2-5)!}$

$$\frac{(x^2-4)\cancel{(x^2-5)}\cancel{(x^2-6)}\cdots}{(x-2)\,\cancel{(x^2-5)}\cancel{(x^2-6)}\cdots}$$

$$\boxed{\frac{x^2-4}{x-2}}$$

$$\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$\boxed{x+2}$$

3. How many 5-digit numbers can be formed with only odd numbered digits? How many of these numbers are bigger than 70,000?

1
3
5
7
9

$$\underline{5}\cdot\underline{5}\cdot\underline{5}\cdot\underline{5}\cdot\underline{5} = 5^5 = \boxed{3125}$$

bigger than 70,000 →

7 or 9

$$\underline{2}\cdot\underline{5}\cdot\underline{5}\cdot\underline{5}\cdot\underline{5} = 2\cdot 5^4 = \boxed{1250}$$

(g) Simplify

$$\frac{17! \cdot \cancel{0!}}{14! \cdot 3!} = \frac{17 \cdot \cancel{16}^{8} \cdot \cancel{15}^{5}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 17 \cdot 8 \cdot 5$$

(h) Compare $\frac{(n+1)!}{(n-1)!}$ with n^2 for any positive integer n:

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1) \cdot n \cdot (n-1) \cdot (n-2) \ldots}{(n-1) \cdot (n-2) \ldots} = (n+1) \cdot n = n^2 + n$$

$$\boxed{\therefore \frac{(n+1)!}{(n-1)!} > n^2}$$

2. Find all possible values of k such that

$$\frac{(2k)!}{(2k-1)!} = \frac{(k+6)!}{(k+5)!}$$

$$\frac{(2k)!}{(2k-1)!} = 2k$$

$$\frac{(k+6)!}{(k+5)!} = k+6$$

$$2k = k + 6$$

$$\boxed{k = 6}$$

3. (a) Last Sunday, at the end of class, all 29 students in the Intermediate group were to form a line at the front of the classroom. In how many ways could they have formed this line?

$$\boxed{29!}$$

(b) At every meeting, Jeff selects 10 students (out of 29) from the class to form the line. He wants to have a different line every time. How many meetings in a row can they continuing forming new lines each time?

$29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$

$$\boxed{\frac{29!}{19!}}$$

Additional Combinatorics Problems

1. There are 10 students in a club. How many ways are there to pick a President and Vice President from the club's members?

$$10 \cdot 9 = \boxed{90}$$

Anna = Pres, Bob = VP $\neq$ Bob = Prez, Ann = VP

2. There are 10 students in another club. How many ways are there to select a committee of two people from the club's members?

$$\frac{10 \cdot 9}{2} = \boxed{45}$$

committee w/ Anna + Bob = committe w/ Bob + Anna

3. How many ways are there to choose two items out of n choices, if the order matters?

$$(n) \cdot (n-1) = \frac{n!}{(n-2)!}$$

4. How many ways are there to choose two items out of n choices, if the order does not matter?

$$\frac{n \cdot (n-1)}{2} = \frac{1}{2!} \cdot \frac{n!}{(n-2)!}$$

5. How many ways are there to choose three items out of n choices, if the order matters?

$$n \cdot (n-1) \cdot (n-2) = \frac{n!}{(n-3)!}$$

6. How many ways are there to choose three items out of n choices, if the order does not matter?

$$\frac{n \cdot (n-1) \cdot (n-2)}{3!} = \frac{1}{3!} \cdot \frac{n!}{(n-3)!}$$

* how many ways are there to choose k items out of n choices?

a) if order does matter?

$$\boxed{\frac{n!}{(n-k)!}}$$

Choose k items from n

b) if order doesn't matter?

$$\frac{1}{\text{duplicates}} \cdot \frac{n!}{(n-k)!}$$

$$\boxed{\frac{1}{k!} \cdot \frac{n!}{(n-k)!}}$$

VOCABULARY ALERT

combinations involve situations where order does not matter

permutations involve situations where order does matter

serendipity is a fancy word that means "good fortune!"

7. There are 20 students in a gym class. How many ways are there to pick a basketball team consisting of 5 people? Is this a combinations problem, or a permutations problem? Did you get the answer right by calculating it correctly, or through serendipity?

combinations.

$$\frac{\cancel{20} \cdot 19 \cdot \overset{6}{\cancel{18}} \cdot 17 \cdot \overset{8}{\cancel{16}}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 19 \cdot 6 \cdot 17 \cdot 8 \rightarrow \boxed{15504}$$

8. HARD QUESTION. There are six letters in the Swiftian language. A word is any sequence of six letters, some pair of which are the same. (The identical letters do not have to be next to each other in the word). How many words are there in this wonderful language?

*on next page

9. Do 7-digit numbers having no digits 1 in their decimal representation constitute more than half of all 7-digit numbers?

10. You are making a bag consisting of 20 pieces of candy to give to your cousin for Halloween and can choose from 8 different types of candy. Assuming you give your cousin at least one of each types of candy, how many different bags could you make?

#8: Swiftian Language

*there are many ways to complete this problem, this is just one of them!

Step 1: assume repeat letter is last letter

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$ repeat

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5$

$6! \cdot 5$

bc only can choose from the 5 letters you're already used before it.

Step 2: account for positions of repeat letter.

repeat letter can go in 6 different spots.

$6! \cdot 5 \cdot 6$

Step 3: account for duplicate words

· for example, if you are repeating the letter a, your repeat letter being in position 3 vs being in position 5 gives you the same word... bc same letter!

$\underline{b}\ \underline{c}\ \underline{a}\ \underline{d}\ \underline{a}\ \underline{e} = \underline{b}\ \underline{c}\ \underline{a}\ \underline{d}\ \underline{a}\ \underline{e}$

so we've double counted.

$$\boxed{\frac{6! \cdot 5 \cdot 6}{2} = 10800}$$