WINTER 2020

OLGA RADKO MATH CIRCLE
ADVANCED 3
MARCH 14, 2020

Each problem is worth 2 points unless otherwise noted.

1. Prime Generating Functions

Problem 1.1. Without citing Dirichlet's Theorem in general, prove that there are infinitely many primes p such that $p \equiv 3 \pmod 4$.

Problem 1.2. Without citing Dirichlet's Theorem in general, prove that there are infinitely many primes p such that $p \equiv 1 \pmod 4$.

(Hint: Assume for contradiction that n is larger than all such primes. Then let p be a prime factor of $(n!)^2 + 1$. What goes wrong?)

2. Borsuk-Ulam

Problem 2.1. I have painted a string of length 1 meter with n colors, coloring the whole string. Show that with n cuts, you can cut this string into $n+1$ pieces such that if you share the pieces with a friend, each of you ends up with half of the total length colored with each color.

(Hint: Assign to each point on S^n a division of the string. That is, for each point $(x_0, \dots, x_n)$ such that $x_0^2 + \cdots + x_n^2 = 1$, describe a placement of n cuts and then an attribution of each of the resulting pieces to either you or your friend. Then find a function $f : S^n \to \mathbb{R}^n$ such that if $x \in S^n$, then $f(x) = f(-x)$ if and only if the division of the string corresponding to x gives you and your friend the same length of each color of string. You don't need to prove that f is continuous.)

Problem 2.2. A particular necklace has on it n kinds of jewels, and an even number of jewels of each kind. Show that it is possible to cut the necklace in n places such that you can divide the resulting $n+1$ pieces between you and a friend and each end up with the same number of each kind of jewel.

3. Dynamics

Problem 3.1. Find a stable periodic point of the logistic map $f_{3.55}$. What period does it have?

Problem 3.2 (Each part is worth 2 points)**.** Let $f : [0,1] \to [0,1]$ be a continuous function, and let x_0 be a periodic point of f with period 7. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$, and so on to $x_6 = f(x_5)$, with $f(x_6) = x_0$. Assume that $x_6 < x_4 < x_2 < x_0 < x_1 < x_3 < x_5$. Use the Itinerary Lemma to show that there are periodic points of f with the following periods:

(1) 2
(2) 4
(3) 6

(4) n for all $n > 7$.

Problem 3.3 (Each part is worth 2 points)**.** When $0 \le h \le 1$, define the *tent map* $T_h : [0,1] \to [0,1]$ by $T_h(x) = \min\left(h, 1 - 2\left|x - \frac{1}{2}\right|\right)$.

(1) What periods do periodic points of T_0 have?
(2) What periods do periodic points of T_1 have?
(3) For what values of h does T_h have a point of period 2?
(4) Show that there is an h for which T_h has a point of period 5 but no point of period 3.

4. 3D Sections

Problem 4.1. Let $ABCD$ be a tetrahedron all of whose edges have length a. The edges DA, DC and BC contain points M, N and P, respectively, so that $|DM| = |CN| = a/4$, $|CP| = a/2$ (here $|\ |$ denotes length). Denote by Q the point at which the plane MNP intersects the edge AB. Find the length of BQ.

Problem 4.2. Let $ABCDA'B'C'D'$ be a parallelepiped (i.e., opposite faces are parallel). Let M, N, P be points on AA', CC' and $C'D'$, respectively, so that $|AM| : |AA'| = |C'N| : |C'C| = |C'P| : |C'D'|$. Find the point Q at which the plane MNP intersects the line BC and find the ratio $|BQ| : |BC|$.

Problem 4.3. Let $ABCD$ be a tetrahedron. A plane is drawn through the vertex C and the midpoints of edges AD and BD. Find the ratio of the parts into which this plane divides the line segment MN, where M and N are the midpoints of the edges AB and CD.

5. P vs. NP

A Hamiltonian Cycle is a path around a graph that touches each vertex once before returning to its starting vertex.

Problem 5.1 (1 point per graph, one attempt)**.** In each of the following graphs, either find a Hamiltonian cycle, or prove none exist. All 8 answers must be submitted together, and you get only one attempt.

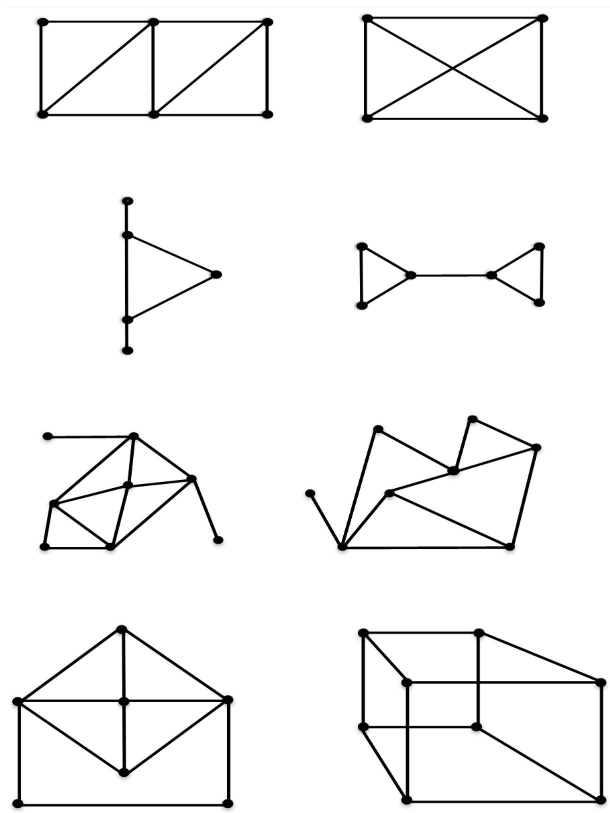
Problem 5.2. Prove Ore's Theorem: Let G be a connected graph with $n \ge 3$ vertices. Assume that for every pair u and v of nonadjacent vertices, $\deg(u) + \deg(v) \ge n$, where $\deg(x)$ is the *degree* of x, that is, the number of edges incident to the vertex x. Show that G has a Hamiltonian cycle.

Problem 5.3. Given that the Hamiltonian Cycle Problem is NP-Complete, show that the Travelling Salesman Problem is also NP-Complete:

A *weighted graph* is a graph G with a function w from the set of edges E to $\mathbb{R}$, such that $w(e) \ge 0$ for all edges e. We call $w(e)$ the *weight* of e, and if P is a path consisting of edges $e_1, e_2, \ldots, e_k$, then define the weight $W(P)$ to be $w(e_1) + w(e_2) + \cdots + w(e_k)$. The Travelling Salesman Problem asks the following: Given a weighted graph (G, w), give a cycle C of minimal weight $W(C)$.

6. Ordinals

Recall from last term that the *Axiom of Choice* says that for every set S and every equivalence relation $\sim$ on S, there is a choice function f for $\sim$ (that is, if $S_\sim$ is the set of equivalence classes of S under $\sim$, then $f : S_\sim \to S$ is such that $f(X) \in X$ for all X).



Problem 6.1. Let S be a set, $\sim$ be an equivalence relation on S, and assume that S is well-ordered. Show that there is a choice function for $\sim$.

Problem 6.2 (Each part is worth 2 points)**.** Assume the Axiom of Choice.

(1) Show that for every set S, there is a function f from the set $P_0(S)$ of nonempty subsets of S to S such that for every nonempty subset $T \subseteq S$, $f(T) \in T$.
(2) Show that every set has can be given a well-ordering.

7. Four Squares Theorem and Gauss Circle Problem

Problem 7.1. In the "orchard" described in the Gauss Circle Problem, set the radius of each tree to 0, so that each tree is represented by just a point. Describe mathematically which trees are visible to a person standing at the center (i.e. not blocked by any other tree).

8. Miscellaneous

Problem 8.1. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all integers written on the 20 faces is 39. Show that there are two faces that share a vertex having the same integer written on them.