THIS IS FUN!

OLGA RADKO MATH CIRCLE
ADVANCED 2
MARCH 14, 2021

1. Graphs

Disclaimer: All graphs in this section are taken to be **simple** graphs. This means that between any two vertices there is at most one edge, and there are no edges between a vertex and itself.

Definition 1. Given a graph G, we define its compliment, denoted $\bar{G}$. $\bar{G}$ is a graph with the same vertex set as G but a different edge set. Two vertices have share an edge in $\bar{G}$ if and only if they do not share an edge in G.

Problem 1. (1) (**5 points**) Let G be the graph with

$$V = \{1, 2, 3, 4, 5\} \text{ and } E = \{(1, 2), (1, 4), (2, 3), (2, 5), (3, 4), (4, 5)\}.$$

Draw G and $\bar{G}$.

(2) (**5 points**) Let G be a graph with n vertices and m edges. How many edges does $\bar{G}$ have?

(3) (**10 points**) Prove that if G is a planar graph with 11 or more vertices then either G or $\bar{G}$ is not planar. (**Hint:** Problem 6 on the planar graphs handout might be useful)

Problem 2. (1) (**10 points**) Prove that if G is disconnected, then $\bar{G}$ is connected. (Being connected means that there is a path between any two vertices)

(2) (**5 points**) Is the converse true? That is, if G is connected does that mean that $\bar{G}$ is disconnected? If true, prove it. If not, give a counterexample.

Problem 3. (**10 points**) Recall the three utilities problem (Problem 2 in Planar graphs worksheet). It is shown in Problem 11 that it has a solution on the torus (check out the solutions). Show that if we add another utility and another house, there is still a solution on the torus. This shows that $K_{4,4}$ is toroidal.

(**Hint:** You can think of the torus as a square with Pac-Man properties. The top side is identified with the bottom, and the left side with the right. So walking off the top side with wrap around to the bottom, and similarly with the left/right sides.)

Problem 4. (**10 points**) Show that the four-color theorem fails on a torus. That is, give a graph on the torus which cannot be colored with only four colors such that no two adjacent vertices share a color.

Definition 2. A graph G is **bipartite** if its vertex set V can be split into two disjoint subsets V_1, V_2 such that there are no edges between vertices in the same subset V_i.

Problem 5. (1) (**5 points**) Draw an example of a bipartite graph with 6 vertices and 9 edges.

(2) (**5 points**) Prove that a graph is 2-colorable if and only if it is bipartite.

2. Continued Fractions

Problem 6. (**20 points**) Write a 9 line poem with the rhyming scheme given by the continued fraction of $\sqrt{91}$.

By this, we mean that each line will be successively assigned a number in the continued fraction and two lines should rhyme if and only if they are assigned the same number. For example, if the continued fraction were $[1, 2, 1, 2, 6, 6]$, then the first line would rhyme with the third, the second line would rhyme with the fourth, and the fifth and sixth lines would rhyme.

Problem 7. (**10 points**) Find a rational number whose rhyming scheme corresponds to that of a limerick.

Problem 8. (**10 points**) Express the continued fraction $[2, \overline{4, 6}]$ in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

Problem 9. (**unlimited points**) It's π day! Compute the best rational approximation for π using a continued fraction. Make sure to simplify and you'll get one point for each digit in the denominator. No cheating!

3. Metrics

Problem 10. (**10 points**) An annulus centered at the origin with inner radius r and outer radius R is the set of all points whose distance to the origin is between r and R. Draw an annulus under the taxicab metric.

Problem 11. (**10 points**) Given a graph G, define the distance between two vertices to be the fewest number of edges needed to travel from one to the other. Viewing the continental US as a graph, where the states are the vertices and adjacent states share an edge, find the ball of radius 2.5 centered at California. (Yes, you can pull up a map)

Definition 3. Let d_E be the Euclidean distance function in $\mathbb{R}^2$. Given two nonempty bounded subsets $X, Y \subset \mathbb{R}^2$ which include their boundaries and a point $x \in X$, let $d(x, Y) = min_{y \in Y}\{d_E(x, y)\}$. This says that the distance from the point x to the subset Y is the distance from x to the closest point in Y. Similarly, define $d(X, y)$ as being the distance from a point $y \in Y$ to the subset X. Now define $d(X, Y)$ to be the maximum of $d(x, Y)$ as you range over all points in X. This means that $d(X, Y)$ is achieved by the point in X which is farthest from Y. Similarly, define $d(Y, X)$ as being the distance achieved by the point in Y which is farthest from X. Finally, let $d_H(X, Y) = max\{d(X, Y), d(Y, X)\}$. We call this the **Hausdorff** distance. This can be thought of as the biggest gap between the two subsets.

Problem 12. (**5 points**) Find the Hausdorff distance between the disk of radius 1 centered at the origin and an inscribed solid square. Draw a picture.

Problem 13. (**10 points**) Prove the first three metric axioms for the Hausdorff distance, d_H. (Keep in mind that your objects are subsets of $\mathbb{R}^2$, not points.)

Problem 14. **(Challenge)**(**20 points**) Prove that d_H satisfies the triangle inequality. That is, show that for any subsets $X, Y, Z \subset \mathbb{R}^2$ we have $d_H(X, Z) \leq d_H(X, Y) + d_H(Y, Z)$.

4. Gini Index

Problem 15. (**10 points**) Find a formula for the three-point estimate of the Gini index. That is, given points $(a, b), (c, d), (e, f)$ with $a < c < e$, what is the Gini index of the curve formed by connecting these points and $(0, 0), (1, 1)$ with straight lines?

Problem 16. (**10 points**) In 2019, the income distribution of households in the United States was as follows:

(1) The top 20% of households earned 51.9% of the total income
(2) The next 20% of households earned 22.7% of the total income
(3) The middle 20% of households earned 14.1% of the total income
(4) The next 20% of households earned 8.3% of the total income
(5) The last 20% earned what's left.

Estimate the household Gini index for the US in 2019.
(In case you were wondering, the real Gini index was 0.48)

Problem 17. (**10 points**) Suppose everyone in a given population is given a non-zero bonus where the amount of the bonus can change from person to person. How should these bonuses be determined so that the Lorenz curve does not change?

5. Competition-Style Problems

Problem 18. (**10 points**) Let $k = 2021^3 + 3^{2021}$. What is the units digit of $k^3 + 3^k$?

Problem 19. (**10 points**) Do there exist 7 circles in the plane such that every circle passes through centers of exactly three other circles? Prove it.

Problem 20. (**10 points**) Do there exist 5 circles in the plane such that every circle passes through centers of exactly three other circles? Prove it.

Problem 21. (**10 points**) Find a sequence of 38 consecutive natural numbers such that none of their digit sums is divisible by 11.

6. Super-Duper Bonus Problem

Important: You must submit all other problems before this one.

Definition 4. Given a graph G, a **Hamiltonian** path through G is a path which goes through every vertex exactly once.

View the continental US as a graph as in Problem 10 and feel free to pretend the borders are somewhat flexible in the sense that if two states touch corners then we consider them adjacent. New York separates the country into two parts, the New England part and the Western part (note that it's not literally the Western part of the country). Since New York is the only bridge between these two parts, any Hamiltonian path through the US will have to start in the New England part (specifically Maine) and end in the Western part, or vice-versa.
Fun fact: Starting in Maine, there exist Hamiltonian paths which end at every single state in the Western part except **one**.

Find the state and prove that there is no Hamiltonian path ending there.