Combinatorics

1. Suppose there are 5 different teacups, 3 different tea saucers, and 4 different teaspoons in the "Tea Party" store. How many ways are there to buy a set consisting of a cup, a saucer, and a spoon?

2. There are four towns A, B, C, and D in Wonderland. In Wonderland, all roads are one-way. Suppose there are 6 different roads from A to B, 3 from A to C, 4 B to C, and 2 C to D. How many ways are there to get from A to C?

3. Re-visit the "Tea Party" store from problem 1. How many ways are there to buy two different items from the set? For example, we could buy one of the cups and one of the spoons, but not two spoons.

4. This time, we have an alphabet with only the letters 'A', 'B', and 'C'. A word in this language is an arbitrary sequence of no more than 4 letters. How many words does the language contain?

5. We have six pieces of fabric, each a different color. How many ways are there to sew a flag with 3 horizontal strips (each of a different color) of equal height? Assume that we can distinguish the top of the flag from the bottom. (What if that were not true?)

6. Same situation, but now we have 200 colors and 40 stripes. What would the math look like in that version of the problem? Do not solve. We like you too much to ask you to actually solve this one to a number. (If we are going to continue doing problems like these, it would be great to have some convenient way to write down patterns of multiplications, wouldn't it?)

If n is a natural number, then n! (pronounced "n factorial") is the product

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

For example, $2! = 2$; $3! = 6$, and $4! = 24$. We also define $0! = 1$.

1. Simplify the following expressions:
   (a)
   $$5! \cdot 6 \cdot 7 =$$

   (b) Express your answer using a factorial:
   $$2 \cdot 3 \cdot 4 =$$

   (c)
   $$\frac{27!}{25!} =$$

   (d) Write this product as a ratio of two factorials:
   $$n \cdot (n-1) = \frac{\qquad !}{\qquad !}$$

   (e) Simplify the fraction:
   $$\frac{n!}{(n-3)!} =$$

   (f) Simplify the fraction
   $$\frac{(2n-1)!}{(2n-3)!} =$$

(g) Simplify

$$\frac{17! \cdot 0!}{14! \cdot 3!} =$$

(h) Compare $\frac{(n+1)!}{(n-1)!}$ with n^2 for any positive integer n:

2. Find all possible values of k such that

$$\frac{(2k)!}{(2k-1)!} = \frac{(k+6)!}{(k+5)!}$$

3. (a) Last Sunday, at the end of class, all 29 students in the Intermediate group were to form a line at the front of the classroom. In how many ways could they have formed this line?

(b) At every meeting, Jeff selects 10 students (out of 29) from the class to form the line. He wants to have a different line every time. How many meetings in a row can they continuing forming new lines each time?

Additional Combinatorics Problems

1. There are 10 students in a club. How many ways are there to pick a President and Vice President from the club's members?

2. There are 10 students in another club. How many ways are there to select a committee of two people from the club's members?

3. How many ways are there to choose two items out of n choices, if the order matters?

4. How many ways are there to choose two items out of n choices, if the order does not matter?

5. How many ways are there to choose three items out of n choices, if the order matters?

6. How many ways are there to choose three items out of n choices, if the order does not matter?

VOCABULARY ALERT

combinations involve situations where order does not matter

permutations involve situations where order does matter

serendipity is a fancy word that means “good fortune!”

7. There are 20 students in a gym class. How many ways are there to pick a basketball team consisting of 5 people? Is this a combinations problem, or a permutations problem? Did you get the answer right by calculating it correctly, or through serendipity?

8. HARD QUESTION. There are six letters in the Swiftian language. A word is any sequence of six letters, some pair of which are the same. (The identical letters do not have to be next to each other in the word). How many words are there in this wonderful language?

9. Do 7-digit numbers having no digits 1 in their decimal representation constitute more than half of all 7-digit numbers?

10. You are making a bag consisting of 20 pieces of candy to give to your cousin for Halloween and can choose from 8 different types of candy. Assuming you give your cousin at least one of each types of candy, how many different bags could you make?