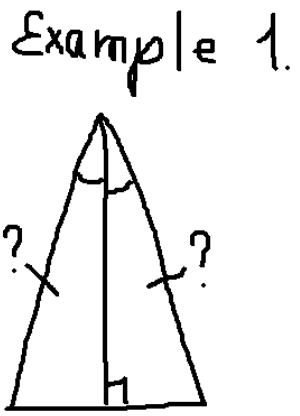
Week 9: Drawing Contest and Invariants

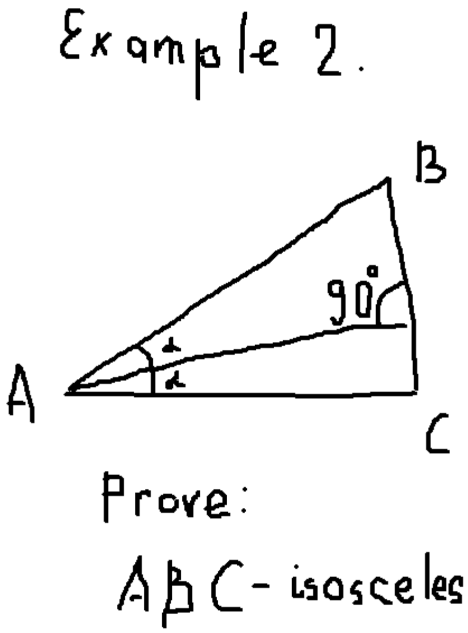
Nikita

1 Drawing contest

Problem 0 (Example).
In a given triangle, an altitude is a bisector. Prove that the triangle is isosceles.

Possible solutions:





You have 30 minutes to draw the pictures illustrating the problems described verbally below. You DON'T need to solve them. Feel free to skip problems you have troubles with. Good luck!

Problem 1.
If two parallel lines are intersected by any line then interior alternate angles are congruent.

Problem 2.
The medians of a triangle are concurrent (they intersect in one common point).

Problem 3.
Lines AB and CD bisect each other. Prove that $AC = BD$.

Problem 4.
In a disk, two chords CC' and DD' perpendicular to a diameter AB are drawn. Prove that the segment MM' joining the midpoints of the chords CD and $C'D'$ is perpendicular to AB.

Problem 5.
From the intersection point of the diagonals of a rhombus, perpendiculars are dropped to the sides of the rhombus. Prove that the feet of these perpendiculars are vertices of a rectangle.

Problem 6.
Consider two rays r, ℓ out of point O, a segment AB on r and point C on ℓ. Let M be the midpoint of AB, let D be the intersection of ℓ and the line through M parallel to AC and let E be the intersection of ℓ and the line through B parallel to AC. Show that D is the midpoint of CE.

Problem 7.
Prove that if through the tangency point of two circles two secants are drawn, then the chords connecting the endpoints of the secants are parallel.

Problem 8.
Prove that if two sides and the median drawn to the first of them in one triangle are respectively congruent to two sides and the median drawn to the first of them in another triangle, then such triangles are congruent.

Problem 9.
Let ABC be a triangle with $\angle C = 90°$, and A_0, B_0, C_0 be the midpoints of sides BC, CA, AB respectively. Two regular triangles AB_0C_1 and BA_0C_2 are constructed outside ABC. Find the angle $C_0C_1C_2$.

2 From previous week

Problem 3.
100 chips are arranged in a row. It is allowed to swap two chips located 2 chips apart (that means you can swap any two chips if there is exactly one chip between them). Is it possible to rearrange all the chips in reverse order using that operation multiple times?

Problem 4.
On a field in the shape of a 10×10 grid 9 squares are infested with weeds. A new square can get infested with weeds if at least two of its adjacent squares are infested. Two squares are called adjacent if they share a side. Show that there will always be a square on the field not infested with weeds. *Hint: Consider the perimeter of the shape infested with weeds.*

3 New problems

Problem 1.
There are 20 scarves hanging on a hanger. 17 girls take turns approaching the hanger, and each either takes off or hangs up exactly one scarf. Can 10 scarves remain on the hanger after the girls leave?

Problem 2.
The changer changes one coin for five others. Is it possible to use it to exchange the coin for 26 coins?

Problem 3.
The numbers 1, 2, 3, ..., 19, 20 are written on the board. It is allowed to erase any two numbers a and b and write the number $a + b - 1$ instead. What number can remain on the board after 19 such operations?