ORMC: COUNTING PROBLEMS

OLYMPIAD GROUP 1, WEEK 7

Problem 1. (Textbook Example)

(a) We have k boxes and n indistinguishable marbles. In how many ways can we distribute the n marbles into the k boxes? (Boxes are allowed to be empty.)

(b) Let n and k be positive integers. In how many ways can we write $n = a_1 + a_2 + \ldots + a_k$, where $a_1, \ldots, a_k$ are *positive* integers?

Problem 2. (Textbook Example) We have an $m \times n$ board, and a cricket on the bottom-leftmost cell $(1, 1)$. The cricket wants to reach the opposite corner (m, n), and it can only do so by jumping one square at a time, either up or to the right. In how many ways can the cricket get to the opposite corner?

Problem 3. We are given a red marbles, b blue marbles and c green marbles. In how many ways can we arrange them in a line? Can you generalize this to n colors (with a_1 marbles of the first color, a_2 of the second color, etc.)?

Problem 4.

(a) Consider a circular necklace consisting of p marbles, where p is a prime number. We want to color each marble in one of a colors. How many different nacklaces can we obtain?

(Two necklaces are considered identical iff you can rotate one to obtain the other; don't worry about flipping the necklaces.)

(b) Conclude that if $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$; this is Fermat's little theorem.

Problem 5. In how many ways can we tile a 2×10 board with dominoes (▯ or ▭)?

Problem 6. We have a 1×20 board, and we want to group the 20 cells into 10 pairs, so that the two cells in each pair are either neighbors (distance 1), or have one other cell in between (distance 2). In how many ways can we do this?

Problem 7. In how many ways can we create a "word" of length 10 with letters A, B or C, such that the sequence AB never appears?

Problem 8.

(a) In how many ways can we color the tiles of a $1 \times n$ board with 3 colors, so that no two adjacent tiles have the same color?

(b) In how many ways can we color the 10 beads of a circular necklace with 3 colors, so that no two adjacent tiles have the same color? (Ignore rotations)

Problem 9. An ant is situated on one of the four cells of a 2×2 board. In how many ways can the ant walk around the board so that she returns to the original square, and makes a total number of 2020 steps (only between adjacent tiles)?

Problem *10. Denote by $\tau(n)$ the number of positive divisors of a positive integer n (so $\tau(1) = 1$, $\tau(4) = 3$, $\tau(6) = 4$, etc.). Show that, for any positive integer n,

$$\tau(1) + \tau(2) + \cdots + \tau(n) = \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor,$$

by counting the number of pairs (a, b) such that $a, b \in \{1, 2, \dots, n\}$ and $a \mid b$, in two ways.

Note: For any real number x, its floor $\lfloor x \rfloor$ is defined as the largest integer less than or equal to x (so e.g., $\lfloor 3 \rfloor = 3, \lfloor 1.5 \rfloor = 1$, $\lfloor -1.2 \rfloor = -2$).

Homework 1

Problem 1. We are given $n \geqslant 6$ white marbles in a line.

Let A be the number of ways to color two of these marbles in red, another two of them in blue, and another two of them in green.

Let B be the number of ways to color six of the initial n white marbles in red. Compute A/B.

Problem 2. In how many ways can we tile a 2×10 board with dominoes and 2×2 blocks (⊟, ◫, or ⊞)?

Homework 2

Problem 1. An ant is situated on one of the 3 beads of a circular necklace. In how many ways can the ant walk around the necklace so that she returns to the original bead, and makes a total number of 10 steps (only between adjacent tiles)?

Problem 2. Find an explicit formula for the recurring sequence defined by

$$a_0 = 0, a_1 = 1, \quad a_n = 3a_{n-1} - 2a_{n-2}.$$

Suggestion: compute small values and try to guess the answer; once you find it, prove it by induction.