

UCLA Math Circle: Paper Folding and Construction I

Yingkun Li

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Using straightedge and compass, it is possible to accomplish many construction, such as construction parallel lines, bisecting arbitrary angle. Some constructions are rather amazing. For example, it is possible to construct regular n -gons with $n = 2^{2^m} + 1$ for some integer m and n a prime number, e.g. $n = 3, 5, 17$. In fact, all the constructions could be done by compass alone according to the Mohr-Mascheroni theorem. On the other hand, several constructions were deemed impossible using straightedge and compass by the Greeks. Among them are the following two problems:

Problem 1 Construct an angle whose measure is $\frac{1}{3}$ of the measure of a given arbitrary angle,

Problem 2 Given a unit length, construct the side length of a cube of volume 2.

In the first problem, we need *arbitrary* since certain special angles can be trisected using straightedge and compass, e.g. a right angle. The second problem is equivalent to construct $\sqrt[3]{2}$. In these notes, which roughly follows [1], we will show that these constructions are possible using origami, i.e. paper folding, and explore the difference between these two types of constructions. First of all, let's see that it is possible to trisect arbitrary angle using origami.

Angle Trisection Procedure

Step 1. Let $\angle AOB$ be an arbitrary acute angle.

Step 2. Fold a line perpendicular to OB through O , call it line ℓ .

Step 3. Fold two lines m_1, m_2 parallel to OB such that they intersect the ray OA and m_2, m_1, OB are evenly spaced in that order. Denote the intersection of ℓ and m_2 by C .

Step 4. Make a fold such that point C is folded onto ray OA and point O is folded onto line m_1 . Denote the intersection of the crease line and line m_1 by D .

Step 5. Fold a line connecting O and D . Then $\angle AOD$ has measure equals to $\frac{1}{3}m\angle AOB$.

Exercise: Prove that the procedure indeed trisects an arbitrary acute angle. Does the spacing between m_1 and m_2 affect the result? How could you modify it for obtuse angles?

After exercises above, you should be convinced that this simple procedure gives a solution to the angle trisection problem. However, an important question arises from it: what are the allowed folding moves in origami construction? For example, in straightedge and compass construction, if one is allowed to make marks on the straight edge and use it to measure, then it is also possible to trisect an arbitrary angle. So specifying the permissible folding moves is important in deciding the range of possibilities. In these notes, we will allow the following folds.

- (F1) Given two points, one can make a fold to identify them, thus creating their perpendicular bisector.
- (F2) Given a line ℓ and a point P , one can create the reflection of P across ℓ .
- (F3) Given a line ℓ and a point P , one can fold a line perpendicular (or parallel) to ℓ and passes through P .
- (F4) Given an arbitrary angle, one can fold its angle bisector
- (F5) Given a line ℓ and points P, Q , then whenever possible, one can fold a line passing through Q and reflect P onto ℓ .
- (F6) Given two lines ℓ, m and points P, Q , then whenever possible, one can make a fold such that P is reflected onto ℓ and Q is reflected onto m .

Using these moves, we saw that it is possible to trisect arbitrary angles. So these allowed moves seem to be more powerful than the usual straightedge compass constructions. To make this intuition precise, we will use axioms to describe the possible constructions. To illustrate the hierarchy of axioms, we will introduce them step by step.

From now on, complex number $a+bi$ represents point in the plane and ℓ_{PQ} denotes a line connecting any two points P, Q in the plane. Let \mathcal{P} be the smallest set of points in the plane and \mathcal{L} the smallest set of lines in the plane such that they satisfy the following axioms

- (A0) There are at least 3 non-colinear points in \mathcal{P} .
- (A1) If $P, Q \in \mathcal{P}$, then $\ell_{PQ} \in \mathcal{L}$.
- (A2) Given non-parallel lines $\ell, m \in \mathcal{L}$, their intersection is in \mathcal{P} .
- (A3) Given two points $P, Q \in \mathcal{P}$, the perpendicular bisector of PQ is in \mathcal{L} .

The idea behind this setup is to begin with a set containing three non-colinear points, which we could always take to be $0, 1, z$ for some complex number z . Then we could gradually build up the set of points and the set of lines connecting them. Eventually, one can reach the sets \mathcal{P} and \mathcal{L} , possibly after applying Zorn's lemma. So \mathcal{P} , resp. \mathcal{L} , corresponds to constructible points, resp. lines. So under these 4 axioms, what are some of the possible points and lines in \mathcal{P} and \mathcal{L} ? The following simple consequence is a good starting point.

Lemma 1. *Any line $\ell \in \mathcal{L}$ contains 2 points in \mathcal{P} . If $P, Q \in \mathcal{P}$, then so is their midpoint.*

Proof. For the first part, since $0, 1, z \in \mathcal{P}$ are non-colinear, they determine three pairwise non-parallel lines. That means ℓ must intersect two of them. By (A2), those two intersections points on ℓ are in \mathcal{P} . The second part follows from (A3), which tells us that the perpendicular bisector of segment PQ , call it ℓ , is contained in \mathcal{L} , and (A2), which says the intersection of ℓ and ℓ_{PQ} , i.e. the midpoint of PQ , is in \mathcal{P} . □

Proposition 1. *Given points $P, A, B \in \mathcal{P}$, then there is a point $E \in \mathcal{P}$ such that AB and PE are parallel, have the same length and direction.*

Proof. First suppose that P is not on ℓ_{AB} . Then we have $\ell_{AP}, \ell_{BP} \in \mathcal{L}$ by (A1), and they are distinct from ℓ_{AB} . Let M, N, R denote the midpoints of AP, BP and AB respectively. By the previous lemma, $M, N, R \in \mathcal{P}$. Now let S be the midpoint of PN , which is also in \mathcal{P} by the same reasoning. Since $M, N, R, S \in \mathcal{P}$, the lines ℓ_{MS} and ℓ_{RN} are both in \mathcal{L} by (A1). Let D be their intersection. Then (A2) tells us that $D \in \mathcal{P}$. In particular, PD is parallel to AB with the same direction and half the length. To construct E , simply replace P with D and repeat exactly the same procedure.

Now suppose P is on the line ℓ_{AB} . Then by (A0), there is another point $P' \in \mathcal{P}$ not on ℓ_{AB} . We can apply the procedure above to construct $E' \in \mathcal{P}$ such that $P'E'$ is parallel to AB in the same direction with the same length. Then applying the same procedure to segment $P'E'$ and point P gives us the desired E . \square

With this proposition, we could move segments and construct parallel lines. To become familiar with arguing using axioms, you can try to prove these exercises.

Exercises:

1. For any $\ell \in \mathcal{L}$ and $P \in \mathcal{P}$, \mathcal{L} contains the line that goes through P and is perpendicular to ℓ .
2. Given a point $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$, then the reflection of P across ℓ is also in \mathcal{P} .
3. Given a segment ABC and an arbitrary point P , we can construct the segment ADP such that $\triangle ABD \sim \triangle ACP$. Consequently, we could divide a segment into arbitrary number of segments of equal lengths. (*Hint: first consider P not on ℓ_{AB}*)

After these exercises, it seems clear that some of the allowed folds mentioned above, such as (F1) - (F3), can be accomplished under the axioms (A0) - (A3). However, it seems difficult to determine whether (F4)-(F6) are also possible under the axioms (A0) - (A3). To answer this question, we will give an algebraic characterization of \mathcal{P} and \mathcal{L} .

Under (A0) above, we have at least three points in \mathcal{P} . Suppose we start with exactly 3 points in \mathcal{P} . Without loss of generality, assume those three points are $0, 1, z$, where z is a complex number with nonzero imaginary part. Denote \mathcal{P}_z the set of points constructed from $0, 1$ and z using (A1) - (A3). So in addition to being a set of points, \mathcal{P}_z is also a set of complex numbers. For a fixed $z = a + ib$, define the following set

$$\begin{aligned} \mathcal{X} &= \{\operatorname{Re}(u) : u \in \mathcal{P}\}, \\ \mathcal{Y} &= \{\operatorname{Im}(u) : u \in \mathcal{P}\}. \end{aligned}$$

Since we could project points onto the x -axis and y -axis, both \mathcal{X} and \mathcal{Y} are subsets of \mathcal{P}_z . However, there are more structures to \mathcal{X} and \mathcal{Y} than just being sets. They are closed under addition, and

are abelian groups. It is less clear what happens under multiplication. The following proposition gives us some ideas.

Proposition 2. *Let $a, b \in \mathcal{X}, u, v, w \in \mathcal{Y}$, all nonzero. Then we have*

1. $a^2, ab \in \mathcal{X}$ and $\frac{1}{a} \in \mathcal{X}$, hence \mathcal{X} is a field.
2. $wv \in \mathcal{X}$ and $aw, \frac{1}{u} \in \mathcal{Y}$.

Following from this proposition, \mathcal{P}_z is a field in the language of abstract algebra. Furthermore it is contained in $\mathbb{Q}(a, b, i)$, the field generated by a, b and i . If both a and b are algebraic, i.e. satisfy polynomial equations with integer coefficients, then the $\mathbb{Q}(a, b, i)$ is in some sense finite over \mathbb{Q} .

To see if (F4) could be carried out via (A0) – (A3), we can characterize it by an axiom first, then study it's relationship to (A0) – (A3). From now on let \mathcal{P}'_z , resp. \mathcal{L}'_z , be the set of points, resp. lines, in the complex plane such that $z \in \mathcal{P}'_z$ and they satisfy (A0) – (A3) and the following axiom

(A4) For any angle formed by two lines $\ell, m \in \mathcal{L}'_z$, its angle bisector is also in \mathcal{L}'_z .

We will call half of a line in \mathcal{L}'_z with starting point in \mathcal{P}'_z a constructed ray, and a segment whose endpoints are both in \mathcal{P}'_z a constructed segment. The following theorem illustrates the power of (A4).

Theorem 1. *Given (A0) - (A3), the followings are equivalent to (A4)*

- (A4') *A unit length segment can be marked on any constructed ray.*
- (A4'') *Any constructed segment can be marked on any constructed ray.*

Proof. We will show that $(A4'') \Rightarrow (A4') \Rightarrow (A4) \Rightarrow (A4'')$. Clearly, (A4'') implies (A4'), since a unit segment is a special constructed segment. For $(A4') \Rightarrow (A4)$, suppose the angle is not 180 degrees, otherwise the angle bisector is just a perpendicular line. We could first mark a unit segment on each of the legs of the angle. Then by the corollaries above, we could construct lines perpendicular to the legs and going through the points. Connecting their intersections with the vertex of the angle gives the angle bisector. Finally for $(A4) \Rightarrow (A4'')$, We could form an angle using the given ray and constructed segment using the first proposition above. Suppose the measure of that angle is not 0 or 180 degrees. After bisecting the angle, we could construct a line going through the endpoint of the constructed segment and perpendicular to the angle bisector. It will intersect the given ray to give the mark. □

From the theorem above, one can see that (A4) is almost equivalent to the compass. With this tool, we can even take square roots of certain numbers, for example $2, 2\sqrt{3} + 5$. One can check that \mathcal{P}'_z is closed under the operation $x \mapsto \sqrt{x^2 + 1}$. That gives us a lot of points in \mathcal{P}'_z . In fact, it is an infinite extension over \mathbb{Q} , in particular, it contains more points than \mathcal{P}_z when z is algebraic. Thus, (A4) cannot be deduced from (A0) – (A3). The numbers obtained from the points 0, 1 and axioms (A1) – (A4) are called Pythagorean numbers.

References

- [1] R. Aperiin, *A mathematical Theory of Origami Constructions and Numbers*, New York J. Math. **6** (2000) 119 - 133