1 Examples of Metrics

Last week, we compared the Euclidean metric, which captures the standard geometric notion of distance, with the taxicab metric which captures a different notion. This is not the only way we can redefine the distance between two things, however; more generally we have metrics.

**Definition 1**  Given a nonempty set $X$, a metric on $X$ is a function $d : X \times X \to \mathbb{R}$ such that

i) $d(x, y) \geq 0$ for all $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

$(X, d)$ is called a metric space.

Last week we saw two notions of distance in the plane $\mathbb{R}^2$ which we called metrics, the Euclidean and taxicab metrics $d_E$ and $d_T$:

$$
\begin{align*}
  d_E((x_1, x_2), (y_1, y_2)) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\
  d_T((x_1, x_2), (y_1, y_2)) &= |x_1 - y_1| + |x_2 - y_2|
\end{align*}
$$

**Problem 1** Verify that $d_E$ and $d_T$ are indeed metrics. (Hint: Most of the hard work was done on last week’s worksheet.)

**Solution**: Students should look to Problems 4 and 5 (parts a and b) from last week’s worksheet, which prove the required properties.

**Problem 2** Define the discrete metric on any nonempty set $X$ by

$$
  d(x, y) = \begin{cases} 
    0 & x = y \\
    1 & x \neq y
  \end{cases}
$$

Prove that $d$ is indeed a metric.

**Solution**: (i) and (ii) follow by definition. (iii) follows from the fact that $x = y$ iff $y = x$ and $x \neq y$ iff $y \neq x$. Finally, for (iv) we note that if $x = z$ then it is trivially satisfied, and if $x \neq z$ then either $x \neq y$ or $y \neq z$ so the right-hand side is also at least 1.

**Problem 3** Let $X$ be the set of six-letter words. Define the Hamming distance between two words as the number of positions in which they differ. For instance, $d(\text{carrot}, \text{potato}) = 6$ and $d(\text{carrot}, \text{carpet}) = 2$. Verify that the Hamming distance is a metric on $X$. 

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General Metrics

Olga Radko Math Circle

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Problem 4  For each function $d$ below, determine (with proof) whether or not $d$ is a metric.

a. $X = \mathbb{N}$ (the positive integers), $d(n, m) = \left| \frac{n}{m} - \frac{m}{n} \right|

b. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|

c. $X$ is the set of vertices of a connected graph $G$, $d(x, y)$ is the minimum number of edges of $G$ needed to connect $x$ to $y$ (where we say zero edges are needed to connect a vertex to itself).

d. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}

e. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \min\{|x_1 - y_1|, |x_2 - y_2|\}

Solution:

a. $d$ is not a metric, since (iv) fails (one counterexample is $n = 2, m = 3, l = 5$)

b. $d$ is not a metric, as (ii) fails, since $d((0, 0), (0, 1)) = 0$ but the two points are not the same (for instance).

c. $d$ is a metric. $d$ is always nonnegative because we cannot have a negative number of edges. (ii) is by definition. For (iii) note that if we have a path from $x$ to $y$ then it is also a path from $y$ to $x$, and vice versa. For (iv) if we have paths from $x$ to $y$ and $y$ to $z$ then concatenating them gives a path from $x$ to $z$.

d. $d$ is a metric. (i) and (iii) follow by definition. For (ii), if the maximum of two nonnegative real numbers is zero then both of them must be zero, so $x_1 = y_1$ and $x_2 = y_2$ and so $x = y$, and conversely if $x = y$ then both of the terms in the maximum are zero. For (iv), we apply the Triangle Inequality on each coordinate:

$|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x, y) + d(y, z)$

$|x_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x, y) + d(y, z)$

so since the inequality holds for each of the two, it holds for their maximum as well.

e. $d$ is not a metric, as (ii) fails, since $d((0, 0), (1, 0)) = 0$ but the two points are not the same (for instance).

2 Balls in Metric Spaces

As we saw last week, one of the most natural things to consider once we have a distance is what the circles look like. We extend this notion to metric spaces in general.

Definition 2 Let $(X, d)$ be a metric space. An open ball of radius $r > 0$ centered at $x_0 \in X$ is the set

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}$$

It is also possible to talk about closed balls $\overline{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\}$, though we will usually stick to open ones.

Problem 5 Draw open balls in $\mathbb{R}^2$ under the Euclidean and taxicab metrics.

Solution: Students should recall the previous week’s worksheet.
Problem 6 Draw or describe the following open balls.

a. $B(0, r)$ ($r$ is a radius $r > 0$) in $X = \mathbb{R}$ with the "standard metric" given by $d(x, y) = |x - y|$.

b. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the metric given by Problem 4d

c. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the discrete metric

d. $B((0, 0), 2)$ in $X = \mathbb{R}^2$ with the discrete metric

e. $B(\text{fly}, 2)$ in $X$ the set of 3-letter words with the Hamming distance

f. $B(4, 3)$ in the path graph $P_{10}$ where the vertices are numbered 1, ..., 10 in order (recall the graph theory handout from a few weeks ago) with the metric given by Problem 4c.

Solution:

a. The open interval $(-r, r)$

b. The inside of the square with the four vertices $(\pm 1, \pm 1)$

c. Just the point $(0, 0)$

d. The entire plane

e. The set of all words "ly", "fy", "fl"

f. The vertices 2, 3, 4, 5, 6