General Metrics

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1 Examples of Metrics

Last week, we compared the Euclidean metric, which captures the standard geometric notion of distance, with the taxicab metric which captures a different notion. This is not the only way we can redefine the distance between two things, however; more generally we have **metrics**.

Definition 1 *Given a nonempty set X, a* ***metric*** *on X is a function $d : X \times X \to \mathbb{R}$ such that*

i) $d(x, y) \geq 0$ *for all* $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$.

iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

(X, d) is called a ***metric space****.*

Last week we saw two notions of distance in the plane $\mathbb{R}^2$ which we called metrics, the Euclidean and taxicab metrics d_E and d_T:

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$
$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Problem 1 *Verify that d_E and d_T are indeed metrics. (Hint: Most of the hard work was done on last week's worksheet.)*

Solution: Students should look to Problems 4 and 5(parts a and b) from last week's worksheet, which prove the required properties.

Problem 2 *Define the* ***discrete metric*** *on any nonempty set X by*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Prove that d is indeed a metric.

Solution: (i) and (ii) follow by definition. (iii) follows from the fact that $x = y$ iff $y = x$ and $x \neq y$ iff $y \neq x$. Finally, for (iv) we note that if $x = z$ then it is trivially satisfied, and if $x \neq z$ then either $x \neq y$ or $y \neq z$ so the right-hand side is also at least 1.

Problem 3 *Let X be the set of six-letter words. Define the* ***Hamming distance*** *between two words as the number of positions in which they differ. For instance, $d(carrot, potato) = 6$ and $d(carrot, carpet) = 2$. Verify that the Hamming distance is a metric on X.*

Solution: The Hamming distance is independent on each position (which can be treated as a coordinate) and is a discrete metric on each coordinate, so argue similarly to Problem 2.

Problem 4 *For each function d below, determine (with proof) whether or not d is a metric.*

a. $X = \mathbb{N}$ *(the positive integers),* $d(n,m) = |\frac{n}{m} - \frac{m}{n}|$

b. $X = \mathbb{R}^2$, $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$

c. *X is the set of vertices of a connected graph G, $d(x,y)$ is the minimum number of edges of G needed to connect x to y (where we say zero edges are needed to connect a vertex to itself).*

d. $X = \mathbb{R}^2$, $d((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

e. $X = \mathbb{R}^2$, $d((x_1,x_2),(y_1,y_2)) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$

Solution:

a. d is not a metric, since (iv) fails (one counterexample is $n = 2$, $m = 3$, $l = 5$)

b. d is not a metric, as (ii) fails, since $d((0,0),(0,1)) = 0$ but the two points are not the same (for instance).

c. d is a metric. d is always nonnegative because we cannot have a negative number of edges. (ii) is by definition. For (iii) note that if we have a path from x to y then it is also a path from y to x, and vice versa. For (iv) if we have paths from x to y and y to z then concatenating them gives a path from x to z.

d. d is a metric. (i) and (iii) follow by definition. For (ii), if the maximum of two nonnegative real numbers is zero then both of them must be zero, so $x_1 = y_1$ and $x_2 = y_2$ and so $x = y$, and conversely if $x = y$ then both of the terms in the maximum are zero. For (iv), we apply the Triangle Inequality on each coordinate:

$$|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x,y) + d(y,z)$$
$$|x_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x,y) + d(y,z)$$

so since the inequality holds for each of the two, it holds for their maximum as well.

e. d is not a metric, as (ii) fails, since $d((0,0),(1,0)) = 0$ but the two points are not the same (for instance).

2 Balls in Metric Spaces

As we saw last week, one of the most natural things to consider once we have a distance is what the circles look like. We extend this notion to metric spaces in general.

Definition 2 *Let (X,d) be a metric space. An **open ball** of radius $r > 0$ centered at $x_0 \in X$ is the set*

$$B(x_0, r) := \{x \in X \; : \; d(x_0, x) < r\}$$

It is also possible to talk about **closed balls** $\overline{B}(x_0, r) := \{x \in X \; : \; d(x_0, x) \leq r\}$, though we will usually stick to open ones.

Problem 5 *Draw open balls in $\mathbb{R}^2$ under the Euclidean and taxicab metrics.*

Solution: Students should recall the previous week's worksheet.

Problem 6 *Draw or describe the following open balls.*

a. $B(0, r)$ *(r is a radius $r > 0$) in $X = \mathbb{R}$ with the ”standard metric” given by* $d(x, y) = |x - y|$.

b. $B((0, 0), 1)$ *in $X = \mathbb{R}^2$ with the metric given by Problem 4d*

c. $B((0, 0), 1)$ *in $X = \mathbb{R}^2$ with the discrete metric*

d. $B((0, 0), 2)$ *in $X = \mathbb{R}^2$ with the discrete metric*

e. $B(fly, 2)$ *in X the set of 3-letter words with the Hamming distance*

f. $B(4, 3)$ *in the path graph P_{10} where the vertices are numbered* $1, ..., 10$ *in order (recall the graph theory handout from a few weeks ago) with the metric given by Problem 4c.*

Solution:

a. The open interval $(-r, r)$

b. The inside of the square with the four vertices $(\pm 1, \pm 1)$

c. Just the point $(0, 0)$

d. The entire plane

e. The set of all words ”_ly”, ”f_y”, ”fl_”

f. The vertices $2, 3, 4, 5, 6$