CONTINUED FRACTIONS

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ADVANCED 2
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It is often necessary to approximate irrational numbers with rational ones. In music theory, for instance, we want the ratio of frequencies in moving an octave to be 2 and a fifth to be 3/2. The equal temperament method of picking note frequencies on a piano requires us to solve an equation like $2^x = \left(\frac{3}{2}\right)^y$. One can show that such. an equation will not have rational solutions so the best we can do is approximate with rational numbers. For more information about this topic, search Pianos and Continued Fractions.

We define a best rational approximation to a real number x as a rational number $\frac{a}{b}$ that is closer to x than any rational number with equal or smaller denominator. The continued fractions process we outline in this worksheet is a method for finding a best rational approximation of any irrational number.

Definition 1. A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}$$

where $a_0, a_1, \ldots, a_k$ are natural numbers. We allow a_0 to be zero while $a_1, \ldots, a_k$ are non-zero. It is also denoted $[a_0, a_1, \ldots, a_k]$.

Problem 1. Write each of the following as a continued fraction:

(a) 5/12
(b) 5/3
(c) 33/23
(d) 37/31

Solution.

(a) $[0, 2, 2, 2]$
(b) $[1, 1, 2]$
(c) $[1, 2, 3, 3]$
(d) $[1, 5, 6]$

Problem 2. Write each of the following continued fractions as a regular fraction in lowest terms:

(a) $[2, 3, 2]$
(b) $[1, 4, 6, 4]$
(c) $[2, 3, 2, 3]$
(d) $[9, 12, 21, 2]$

Solution.

(a) 16/7
(b) 129/104

(c) 55/24
(d) 4705/518

Problem 3. Let p/q be a positive rational number in lowest terms. Perform the Euclidean algorithm to obtain the following sequence:

$$\begin{aligned} p &= q_0 q + r_1 \\ q &= q_1 r_1 + r_2 \\ r_1 &= q_2 r_2 + r_3 \\ &\vdots \\ r_{k-1} &= q_k r_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$

(We know that we will eventually get 1 as the remainder because p and q are relatively prime). Prove that $p/q = [q_0, q_1, \ldots, q_{k+1}]$.

Solution. Divide both sides of the first equation above by q. Divide both sides of the ith equation in the Euclidean algorithm by r_{i-1} for $i \geq 2$. Then

$$\begin{aligned} \frac{p}{q} &= q_0 + \frac{r_1}{q} \\ &= q_0 + \frac{1}{\frac{q}{r_1}} \\ &= q_0 + \frac{1}{q_1 + \frac{r_2}{r_1}} \\ &\vdots \\ &= q_0 + \cfrac{1}{q_1 + \ldots + \cfrac{1}{q_{k-1} + \frac{r_k}{r_{k-1}}}} \\ &= q_0 + \cfrac{1}{q_1 + \ldots + \cfrac{1}{q_k + \cfrac{1}{q_{k+1}}}} \end{aligned}$$

Problem 4. Repeat Problem 1 using the method outlined in Problem 3.

Solution.
(a) $[0, 2, 2, 2]$
(b) $[1, 1, 2]$
(c) $[1, 2, 3, 3]$
(d) $[1, 5, 6]$

Definition 2. Continued fractions don't always have to be finite. An *infinite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}}$$

where $a_0, a_1, a_2, \dots$ are natural numbers. To prove that this expression actually makes sense and equals a finite number is beyond the scope of this worksheet, so we assume it for now. This is denoted $[a_0, a_1, a_2, \dots]$. Infinite continued fractions always represent irrational numbers, and every irrational number has an expression as a continued fraction. Unfortunately, these claims are also beyond the scope of the worksheet.

Problem 5. Using a calculator, compute the first five terms of the continued fraction expansion of the following numbers:

(a) $\sqrt{2}$
(b) π ($\approx 3.14159\dots$)
(c) $\sqrt{5}$
(d) e ($\approx 2.71828\dots$)

Do you notice any patterns?

Solution. Take the irrational number, subtract the integer part, and reciprocate. Repeat this process. The integer parts are the natural numbers in the continued fraction.

(a) $\sqrt{2} \approx [1, 2, 2, 2, 2]$
(b) $\pi \approx [3, 7, 15, 1, 292]$
(c) $\sqrt{5} \approx [2, 4, 4, 4, 4]$
(d) $e \approx [2, 1, 2, 1, 1]$

Problem 6. Let α be a positive real number. Prove that α can be written as a finite continued fraction if and only if α is rational. (Hint: For one of the directions, use Problem 3.)

Solution. ($\Rightarrow$) Assume that $\alpha = [a_0, \dots, a_k]$. We can combine $a_{k-1} + \frac{1}{a_k} = \frac{a_{k-1}a_k+1}{a_k}$. Then we can combine a_{k-2} and the reciprocal of the previous term via $a_{k-2} + \frac{a_k}{a_{k-1}a_k+1} = \frac{a_{k-2}a_{k-1}a_k+a_{k-2}+a_k}{a_{k-1}a_k+1}$. Continuing this process finitely many times, we find that α is rational.

($\Leftarrow$) Assume that α is rational. Then $\alpha = \frac{p}{q}$ for some relatively prime natural numbers p and q. The Euclidean algorithm when p is divided by q will terminate in finitely many steps with quotients $q_0, q_1, \dots, q_{k+1}$. Thus Problem 3 implies $\frac{p}{q} = [q_0, q_1, \dots, q_{k+1}]$.

Definition 3. The continued fraction $[a_0, a_1, a_2, \dots]$ is *periodic* if it is of the form

$$[a_0, \dots, a_r, a_{r+1}, \dots, a_{r+p}, a_{r+1}, \dots, a_{r+p}, a_{r+1}, \dots, a_{r+p}, \dots].$$

In this case it is denoted $[a_0, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+p}}]$.

Example 1. (a) $[1, 2, 2, 2, \dots] = [1, \overline{2}]$ is periodic.
(b) $[1, 2, 3, 4, 5, \dots]$ is *not* periodic.
(c) $[1, 3, 7, 6, 4, 3, 4, 3, 4, 3, \dots] = [1, 3, 7, 6, \overline{4, 3}]$ is periodic.
(d) $[1, 2, 4, 8, 16, \dots]$ is *not* periodic.

Problem 7. (a) Prove that $\sqrt{2} = [1, \overline{2}]$.
(b) Prove that $\sqrt{5} = [2, \overline{4}]$.

Solution.

(a) We have $\sqrt{2} = 1+(\sqrt{2}-1)$. Then the integer part of $\frac{1}{\sqrt{2}-1}$ is 2. Rationalizing the denominator of the fraction, we have $\frac{1}{\sqrt{2}-1}-2 = (\sqrt{2}+1)-2 = \sqrt{2}-1$. The process will repeat so $\sqrt{2} = [1, \overline{2}]$.

(b) We have $\sqrt{5} = 2+(\sqrt{5}-2)$. Then the integer part of $\frac{1}{\sqrt{5}-2}$ is 4. Rationalizing the denominator of the fraction, we have $\frac{1}{\sqrt{5}-2}-4 = (\sqrt{5}+2)-4 = \sqrt{5}-2$. The process will repeat so $\sqrt{5} = [2, \overline{4}]$.

Problem 8. Express the following continued fractions in the form $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers:

(a) $[\overline{1}]$
(b) $[\overline{2, 5}]$
(c) $[1, \overline{3, 2}]$

Solution.

(a) The continued fraction α satisfies $\alpha = \frac{1}{\alpha-1}$. Thus $\alpha^2 - \alpha = 1$. The quadratic formula gives $\alpha = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$. We take the positive choice for α.

(b) The continued fraction α satisfies $\alpha = \frac{1}{\frac{1}{\alpha-2}-5} = \frac{\alpha-2}{-5\alpha+11}$. Thus $-5\alpha^2 + 10\alpha + 2 = 0$. The quadratic formula gives $\alpha = \frac{-10\pm\sqrt{100+40}}{-10} = \frac{5\pm\sqrt{35}}{5}$. We take the positive choice for α.

(c) We will first find an expression for $\beta = [\overline{3, 2}]$. The continued fraction satisfies $\beta = \frac{1}{\frac{1}{\beta-3}-2} = \frac{\beta-3}{-2\beta+7}$. Thus $-2\beta^2 + 6\beta + 3 = 0$. The quadratic formula gives $\beta = \frac{-6\pm\sqrt{36+24}}{-4} = \frac{3\pm\sqrt{15}}{2}$. We take the positive choice for β. Then $\alpha = 1 + \frac{1}{\beta} = \frac{5+\sqrt{15}}{3+\sqrt{15}}$. Rationalizing the denominator and simplifying, we obtain $\alpha = \frac{\sqrt{15}}{3}$.

Problem 9. Prove that a number is rational if and only if it is a root of a degree one polynomial equation with integer coefficients.

Solution. ($\Rightarrow$) Let $\frac{p}{q}$ be a rational number. Let $A = q$ and $B = -p$ so it satisfies $Ax + B = 0$.
($\Leftarrow$) Let α satisfy $Ax + B = 0$ for integers A and B. Then $\alpha = \frac{-B}{A}$ is rational.

Problem 10. Prove that an irrational number is of the form $\frac{a+\sqrt{b}}{c}$ for integers a, b not a perfect square, and $c \neq 0$ if and only if it is a root of a degree two polynomial equation with integer coefficients.

Solution. ($\Rightarrow$) Let $A = c^2$, $B = -2ac$, and $C = a^2 - b$. Then $\frac{a+\sqrt{b}}{c}$ satisfies $Ax^2 + Bx + C = 0$ where A, B, and C are integers. Since $c \neq 0$, we have $A \neq 0$.

($\Leftarrow$) Assume that α irrational is a root of the polynomial $Ax^2 + Bx + C = 0$ for integers $A \neq 0$, B, and C. Then $\alpha = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$ by the quadratic formula. Since $A \neq 0$, the denominator $2A \neq 0$. Further, $B^2 - 4AC \leq 0$ or $B^2 - 4AC$ a perfect square implies that α is complex or rational. We assume α is irrational. Thus $B^2 - 4AC > 0$ is not a perfect square and $\alpha = \frac{-B+\sqrt{B^2-4AC}}{2A}$ as desired.

Definition 4. A *quadratic irrational* is an irrational number that is a solution to a quadratic equation $Ax^2 + Bx + C = 0$ for integers $A \neq 0$, B, and C.

We show in Problem 10 that quadratic irrationals are numbers of the form $\frac{a+\sqrt{b}}{c}$ where a, b, c are integers and b is not a perfect square. They are the "simplest" irrational numbers in the following sense. A number is rational if and only if it is the solution to a degree one polynomial equation by Problem 9. Quadratic irrationals are solutions to degree two polynomial equations.

Problem 11 (Challenge)**.** We will show that a periodic continued fraction is a quadratic irrational.

(a) Prove that $a_0 + \frac{1}{a_1+\cdots+\frac{1}{a_k+\frac{1}{x}}}$ can be written in the form $\frac{ax+b}{cx+d}$ for some integers a, b, c, and d.

(b) Prove that a purely periodic continued fraction $[\overline{a_0, \ldots, a_k}]$ is a quadratic irrational.

(c) Prove, by induction, that $[b_0, \ldots, b_m, \overline{a_0, \ldots, a_k}]$ is a quadratic irrational.

Solution.

(a) We can combine $a_k + \frac{1}{x}$ to obtain $\frac{a_k x+1}{x}$. Then $a_{k-1} + \frac{x}{a_k x+1} = \frac{(a_{k-1}a_k+1)x+a_{k-1}}{a_k x+1}$. Repeating this process, we obtain $\frac{ax+b}{cx+d}$ for integers a, b, c, and d.

(b) Let $x = [\overline{a_0, \ldots, a_k}]$. Since x is an infinite continued fraction, it is irrational. We can write $x = [\overline{a_0, \ldots, a_k}] = a_0 + \frac{1}{a_1+\cdots+\frac{1}{a_k+\frac{1}{x}}}$. By part (a), we have $x = \frac{ax+b}{cx+d}$ for some integers a, b, c, and d. Then $cx^2 + (d-a)x - b = 0$. We note that $c \neq 0$ by x irrational. Thus x is quadratic irrational.

(c) We will first show that $b_0 + \frac{1}{x}$ is quadratic irrational when x is quadratic irrational. Write $x = \frac{a+\sqrt{b}}{c}$ for integers a, b not a perfect square, and $c \neq 0$ by Problem 10. Then $b_0 + \frac{1}{x} = \frac{(b_0a^2+ac-b_0b)-c\sqrt{b}}{a^2-b}$ is quadratic irrational since $a^2 - b \neq 0$ for b not a perfect square.

Assume that $[b_1, \ldots, b_m, \overline{a_0, \ldots, a_k}]$ is a quadratic irrational. Then $[b_0, \ldots, b_m, \overline{a_0, \ldots, a_k}]$ is $b_0 + \frac{1}{[b_1,\ldots,b_m,\overline{a_0,\ldots,a_k}]}$. By the inductive hypothesis, $[b_1, \ldots, b_m, \overline{a_0, \ldots, a_k}]$ is quadratic irrational. The argument above completes the proof.

One can also prove that a quadratic irrational can always be written as a periodic continued fraction. The results of this worksheet provide a very clean characterization of continued fraction expansions:

(a) α is a rational number if and only if it has a finite continued fraction expansion.

(b) α is a quadratic irrational number if and only if it has an infinite periodic continued fraction expansion.