General Metrics

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1 Examples of Metrics

Last week, we compared the Euclidean metric, which captures the standard geometric notion of distance, with the taxicab metric which captures a different notion. This is not the only way we can redefine the distance between two things, however; more generally we have **metrics**.

Definition 1 *Given a nonempty set X, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that*

i) $d(x, y) \geq 0$ *for all* $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$.

iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

*(X, d) is called a **metric space**.*

Last week we saw two notions of distance in the plane $\mathbb{R}^2$ which we called metrics, the Euclidean and taxicab metrics d_E and d_T:

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$
$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Problem 1 *Verify that d_E and d_T are indeed metrics. (Hint: Most of the hard work was done on last week's worksheet.)*

Problem 2 *Define the **discrete metric** on any nonempty set X by*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Prove that d is indeed a metric.

Problem 3 *Let X be the set of six-letter words. Define the **Hamming distance** between two words as the number of positions in which they differ. For instance, $d(carrot, potato) = 6$ and $d(carrot, carpet) = 2$. Verify that the Hamming distance is a metric on X.*

Problem 4 *For each function d below, determine (with proof) whether or not d is a metric.*

a. $X = \mathbb{N}$ *(the positive integers),* $d(n, m) = |\frac{n}{m} - \frac{m}{n}|$

b. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|$

c. *X is the set of vertices of a connected graph G, $d(x, y)$ is the minimum number of edges of G needed to connect x to y (where we say zero edges are needed to connect a vertex to itself).*

d. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

e. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$

2 Balls in Metric Spaces

As we saw last week, one of the most natural things to consider once we have a distance is what the circles look like. We extend this notion to metric spaces in general.

Definition 2 *Let (X, d) be a metric space. An **open ball** of radius $r > 0$ centered at $x_0 \in X$ is the set*

$$B(x_0, r) := \{x \in X \ : \ d(x_0, x) < r\}$$

It is also possible to talk about **closed balls** $\overline{B}(x_0, r) := \{x \in X \ : \ d(x_0, x) \leq r\}$, though we will usually stick to open ones.

Problem 5 *Draw open balls in $\mathbb{R}^2$ under the Euclidean and taxicab metrics.*

Problem 6 *Draw or describe the following open balls.*

a. *$B(0, r)$ (r is a radius $r > 0$) in $X = \mathbb{R}$ with the "standard metric" given by $d(x, y) = |x - y|$.*

b. *$B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the metric given by Problem 4d*

c. *$B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the discrete metric*

d. *$B((0, 0), 2)$ in $X = \mathbb{R}^2$ with the discrete metric*

e. *$B(fly, 2)$ in X the set of 3-letter words with the Hamming distance*

f. *$B(4, 3)$ in the path graph P_{10} where the vertices are numbered $1, ..., 10$ in order (recall the graph theory handout from a few weeks ago) with the metric given by Problem 4c.*

3 Convergence of Sequences

Definition 3 *A **sequence** in a set X is a (countably) infinite collection $x_1, x_2, x_3, ...$ where $x_n \in X$ for all $n \in \mathbb{N}$. Denote it $(x_n)_{n=1}^{\infty}$.*

As an example, we consider the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, where

$$x_n = \frac{1}{n}$$
$$y_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

in $\mathbb{R}$ with the standard metric.

Problem 7 *a. Show that any open ball centered at* 0 *contains all but finitely many elements of* x_n.

b. Show that even though any open ball centered at 0 *contains infinitely many elements of* y_n*, it may not contain all but finitely many elements of* y_n.

Definition 4 *A sequence* $(x_n)_{n=1}^{\infty}$ *in a metric space* X ***converges*** *to* $x \in X$ *if every open ball centered at* x *contains all but finitely many elements of the sequence. We also say that* x *is a* ***limit*** *of the sequence* $(x_n)_{n=1}^{\infty}$ *and we write* $x_n \to x$ *or* $x = \lim_{n\to\infty} x_n$ *(This limit is not a priori unique, but we will see later that it is.)* †

By Problem 7 we see that in the previous example, $x_n \to 0$, but $y_n \nrightarrow 0$.

Problem 8 *Consider the sequence* $(x_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^2$*, where* $x_n = (1/n, -1/n)$.

a. Does this sequence converge under the Euclidean metric d_E*? If so, to what limit?*

b. Does this sequence converge under the taxicab metric d_T*? If so, to what limit?*

c. Does this sequence converge under the discrete metric? If so, to what limit?

d. Does this sequence converge under the metric d *given by Problem 4d? If so, to what limit?*

Problem 9 *Let* X *be the metric space of* 3*-letter words with the Hamming distance.*

a. Does the sequence $fly, fly, fly, fly, fly...$ *converge? If so, to what limit?*

b. Does the sequence $sky, sly, fly, fly, fly...$ *converge? If so, to what limit?*

c. Does the sequence $sly, fly, sly, fly, sly...$ *converge? If so, to what limit?*

Definition 5 *A* ***subsequence*** *of* $(x_n)_{n=1}^{\infty}$ *is an infinite sequence consisting of the same elements* $x_1, x_2, ...$*, except possibly skipping some.*

Continuing the previous example, we see that the sequence $1, 1/2, 1/4, 1/8, ... = (2^{-n})_{n=1}^{\infty}$ is a subsequence of $(1/n)_{n=1}^{\infty}$, because all the elements are $1/n$ for some n but some elements (those that are not powers of 2) are skipped. Again take any open ball around zero. If its radius is r, then eventually we have $2^n > 1/r$ so $-r < 0 < 2^{-n} < r$ and again all but finitely many elements lie in the open ball $B(0, r)$ so that $2^{-n} \to 0$ as well.

Problem 10 *(Challenge)*

a. Prove that if $x_n \to x$*, then any subsequence of* $(x_n)_{n=1}^{\infty}$ *also converges to* x.

b. Prove that a sequence cannot converge to two different points (and thus we are justified in writing $x = \lim_{n\to\infty} x_n$*).*

†Some astute observers may notice that this definition of convergence appears different from what one would typically encounter in, say, a calculus textbook. It is a good mathematical exercise to figure out, at least in the case with the standard metric on $\mathbb{R}$, why this definition is equivalent to the definition that $x_n \to x$ if for any $\epsilon > 0$, $|x_n - x| < \epsilon$ for all n sufficiently large.