Goodstein Sequences and Ordinals

Adapted from a worksheet by Kyle Hess

February 28, 2021

1 Goodstein sequences

Let b, n be natural numbers. (As an example, we'll try $b = 3$ and $n = 100$.) To write n in *hereditary base b notation*, first write n in base b, like so: $100 = 3^4 \cdot 1 + 3^2 \cdot 2 + 3^0 \cdot 1$. Each term should be of the form $b^k \cdot c$, where c is a coefficient less than b, and k is some natural number. Now to make this *hereditary* base b notation, we also write the exponents in base b, and so on! In this case, we only need to change the one exponent that's bigger than the base, $b = 3$, as everything else can be written as $c \cdot b^0$. Thus we get

$$100 = 3^{3^1 \cdot 1 + 3^0 \cdot 1} \cdot 1 + 3^2 \cdot 2 + 3^0 \cdot 1.$$

If we want to save a little bit of time on this time-intensive process, we can skip writing things like $x \cdot 1$ instead of x, x^1 instead of x, and x^0 instead of 1, and write it like this:

$$100 = 3^{3+1} + 3^2 \cdot 2 + 1.$$

This is in hereditary base 3 notation, because all of the numbers that appear are at most 3. To get to hereditary base b notation, keep replacing exponents with their base b form until you run out of numbers greater than b!

PROBLEM 1. Write 35 in hereditary base 2 notation.

To define the *Goodstein sequence* starting with a number m, first let $G(m)(2) = m$. Then to calculate $G(m)(3)$, write m in hereditary base 2 notation, and replace every 2 with a 3, and then subtract 1.

PROBLEM 2. If $m = 35$, what is $G(35)(3)$? How many digits does it have?

Then to calculate $G(m)(4)$, we write $G(m)(3)$ in hereditary base 3 notation, replace all the 3s with 4s, and then subtract 1, and so on. Whenever you have $G(m)(b)$, write it in hereditary base b notation, replace all the bs with $b + 1$, and then subtract 1. Then the Goodstein sequence is $G(m)(2), G(m)(3), G(m)(4), G(m)(5), \ldots$.

Problem 3. Calculate as many values as you can of the Goodstein sequence starting at 3. Does this sequence grow quickly?

It turns out that not all Goodstein sequences grow as slowly as $G(3)$. You've seen that $G(35)$ immediately rockets up from 2 digits at $G(35)(2)$ to many more for $G(35)(3)$, and by the time you get to, say, $G(35)(5)$, it is absolutely enormous! However, using the powerful tools of *ordinals*, we can show that all Goodstein sequences, no matter how quickly they grow at first are like $G(3)$, and eventually fall down to 0.

2 Introduction to Ordinals

The *ordinals* are a generalization of the natural numbers.

We can think of the natural numbers as being the smallest ordered collection of objects satisfying the following rules:

1. There is a natural number, 0, which is smaller than any other natural number.
2. If n is a natural number, then there is a natural number $S(n)$ (called the *successor* of n) such that $S(n) > n$ and there are no natural numbers between n and $S(n)$.

We can think of the above rules as "construction rules". The first one allows us to "construct" the natural number 0, and the second one allows us to construct every other natural number from 0. For example, we can think of 1 as just "S(0)", two as just "S(S(0))", and so on.

Problem 4. Construct the number 5 using just 0 and the function S.

If we add a third construction rule, we get an even larger number system, called the *ordinals*:

3. For every set of ordinals X, there is a smallest ordinal strictly greater than every element of X.

Using this new rule we can, for example, construct a new ordinal ω which is defined as the least ordinal strictly greater than any natural number. We call ω the *least strict upper bound* of the natural numbers, and denote this by writing $\omega = \text{lsub}(\{0, S(0), S(S(0)), \ldots\})$

Problem 5. Show that rules 1 and 2 are actually redundant once we add rule 3; that is, use rule 3 to show that the ordinals also have a smallest element, which we will call 0, and a successor function S.

Using rule 2, we can actually construct ordinals even larger than ω, such as $S(\omega), S(S(\omega))$, and so on. We can even form an ordinal called "$\omega \cdot 2$" which is equal to

$$\text{lsub}(\{\omega, S(\omega), S(S(\omega)) \ldots\})$$

From here, we can similarly define $\omega \cdot 3$ as

$$\text{lsub}(\{\omega \cdot 2, S(\omega \cdot 2), S(S(\omega \cdot 2)), \ldots\})$$

And in general we can define $\omega \cdot (n+1)$ in terms of $\omega \cdot n$ as

$$\text{lsub}(\{\omega \cdot n, S(\omega \cdot n), S(S(\omega \cdot n)), \ldots\})$$

We could keep going, but first let's try to justify some of this notation. $\omega \cdot 2$ makes it seem like we are multiplying ω by 2, but we never defined multiplication for ordinals! Sure, we know what it means to multiply two natural numbers, but what does it mean to multiply two ordinals, especially ones that aren't natural numbers? To define this, we'll need the concept of a well-ordered set:

Definition 6. An ordered collection C is *well-ordered* if any nonempty set $S \subset C$ has a smallest element.

An example of a set that is well-ordered would be the natural numbers $\mathbb{N}$. An example of a set that isn't well-ordered would be the set of all nonnegative real numbers, since there is a subset of it (the positive reals) which doesn't have a smallest element (i.e.- there is no smallest positive real number).

Problem 7. Show that the ordinals are well-ordered.

Note that for technical reasons, we can't call the ordinals a well-ordered set... because it's too large of a collection to be a set.

Problem 8 (Challenge). Assume that there is a set S of all ordinals. Find a contradiction.

Problem 9. Show that any subcollection of a well-ordered collection is well-ordered.

These two problems together imply that any set of ordinals is a well-ordered set. As it turns out, the converse is also true: any well-ordered set can be thought of as a set of ordinals!

Theorem 10. *For any well-ordered set S, there is a set of ordinals M such that there is a function $f : S \to M$ satisfying the following two properties:*

1. *f is one-to-one correspondence between S and M; that is, for every element $y \in M$ there is exactly one $x \in S$ such that $f(x) = y$.*
2. *f preserves the ordering on S; that is, $x, y \in S$ satisfy $x < y$ if and only if $f(x) < f(y)$.*

This theorem allows us to think of sets of ordinals and well-ordered sets as essentially the same things. In fact, there is an even stronger claim we can make using the following fact:

Theorem 11. *Let $ORD_{<\alpha}$ be the set of all ordinals strictly less than some ordinal α. Then for any set of ordinals M, there is a unique α such that there exists $f : M \to ORD_{<\alpha}$ satisfying the properties of Theorem 10.*

Theorem 11 strengthens the equivalence of Theorem 10 by allowing us to only consider so-called "initial segments" of ordinals, or those of the form $ORD_{<\alpha}$ for some α. Since there is a correspondence between ordinals α and initial segments $ORD_{<\alpha}$, we can think of ordinals themselves as being the same as well-ordered sets, and vice versa. Using this equivalence, if we define addition and multiplication on well-ordered sets, we can easily translate this into a definition on ordinals! So how do we define addition and multiplication on well-ordered sets?

Definition 12. The sum of two well-ordered sets A, B is the disjoint union $A \sqcup B$ with the ordering $x < y$ if and only if one of the following is true:

1. $x, y \in A$ and $x < y$ in A
2. $x, y \in B$ and $x < y$ in B
3. $x \in A$, $y \in B$

You can think of the sum of two well-ordered sets as the result of putting them right next to each other, with all the elements of one strictly greater than all the elements of the other. The next problem asks you to show that this definition is actually valid.

Problem 13. Show that the ordering we defined on $A \sqcup B$ is actually a well-ordering. That is, show that:

1. It is an ordering. That is:
   (a) It is never true that $x < x$ for any $x \in A \sqcup B$
   (b) Exactly one of the following is true for any $x, y \in A \sqcup B$: $x < y$, $y < x$, or $x = y$
   (c) If $x, y, z \in A \sqcup B$ satisfy $x < y$ and $y < z$, then they must also satisfy $x < z$
2. It is additionally a *well*-ordering. That is, any subset $S \subset A \sqcup B$ has a smallest element.

Now we can define addition on ordinals as follows:

Definition 14. Let α, β be ordinals. Then define $\alpha + \beta$ to be the unique ordinal such that there is an order-preserving one-to-one correspondence $f : \mathrm{ORD}_{<\alpha} \sqcup \mathrm{ORD}_{<\beta} \to \mathrm{ORD}_{<\alpha+\beta}$.

Next, we define multiplication on well-ordered sets, and then on ordinals:

DEFINITION 15. The product of two well-ordered sets A, B is their cartesian product $A \times B$ with the lexicographic ordering $x < y$ if and only if one of the following is true:

1. $x = (a, b), y = (a', b')$ with $a < a'$
2. $x = (a, b), y = (a', b')$ with $a = a'$ and $b < b'$

Just as there is an intuitive way to think of ordinal addition, there is an intuitive way to think of ordinal multiplication. To do this, we think of replacing each element of A with a copy of B, such that each element from an earlier copy of B is less than any element from a later copy and two elements within the same copy are ordered according to their standard ordering inside B.

PROBLEM 16. Go through the same steps as in problem 8 to show that the product of two well-ordered sets is actually a well-ordering.

DEFINITION 17. Let α, β be ordinals. Then define $\alpha \cdot \beta$ to be the unique ordinal such that there is an order-preserving one-to-one correspondence $f : \mathrm{ORD}_{<\alpha} \times \mathrm{ORD}_{<\beta} \to \mathrm{ORD}_{<\alpha\cdot\beta}$.

PROBLEM 18. For any ordinal α, show that $S(\alpha) = \alpha + 1$. That is, $\alpha + 1$ is the smallest ordinal strictly greater than α. Note that this does not follow immediately from the definitions!

PROBLEM 19. Show that $1 + \omega = \omega$.

Problems 18 and 19 together imply that addition of ordinals is noncommutative, since $1 + \omega = \omega \neq S(\omega) = \omega + 1$. However, as you become experienced with ordinal addition and multiplication, you will find patterns that make thinking about these concepts more intuitive.

PROBLEM 20. Show that ordinal multiplication is also noncommutative.

One of the patterns that makes thinking about these concepts easier is the fact that, at the very least, ordinal addition and multiplication are associative.

PROBLEM 21. Show that ordinal addition and multiplication are associative. That is, for all ordinals α, β, γ it is true that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

This means we can say things like "ω multiplied by itself n times" without being ambiguous (if multiplication wasn't associative, it might not be clear *in what order* to evaluate the multiplications). In fact, the previous statement is important enough that we have a special notation for it: ω^n. We can also recursively define this as $\omega^0 = 1, \omega^{n+1} =$

$\omega^n \cdot \omega$. Note that associativity tells us that we could have also defined $\omega^0 = 1, \omega^{n+1} = \omega \cdot \omega^n$ and we would have the same value for ω^n.

Problem 22. Show that if $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ are both finite sequence of finite ordinals (i.e.- natural numbers), then the following is true...

$$\omega^n \cdot a_n + \cdots + \omega^1 \cdot a_1 + \omega^0 \cdot a_0 > \omega^n \cdot b_n + \cdots + \omega^1 \cdot b_1 + \omega^0 \cdot b_0$$

...if and only if there is an i such that $a_i \neq b_i$ and the greatest such i satisfies $a_i > b_i$. That is, to compare two ordinals of the forms above, we compare the "coefficients" of the highest exponents of ω until we come across two that are unequal. (You may assume that for all ordinals α, β, γ, then $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.)

3 Goodstein Sequences and Ordinals

Now we can return to Goodstein sequences.

Before we show that Goodstein sequences terminate, we'll show that a similar but slightly simpler kind of sequence terminates.

Given a starting number m, let $g(m)(2) = m$. To calculate $g(m)(3)$, instead of writing $g(m)(2)$ in *hereditary* base 2 notation, just write it in base 2. For instance, we would write $15 = 2^3 + 2^2 + 2^1 + 2^0$. Then without changing the exponents, replace the 2s with 3s, and then subtract one. In our example, we get $g(15)(3) = 3^3 + 3^2 + 3^1 + 3^0 - 1 = 39$. Then we continue, writing $g(m)(b)$ in non-hereditary base b, replacing the non-exponent bs with $b + 1$, and then subtract 1.

Problem 23. Show that there is some b for which $g(15)(b) > 1,000,000$.

Given numbers m, b, with $2 \leq b$, define the ordinal $g'(m)(b)$ by writing $g(m)(b)$ in base b notation, and replacing every non-exponent instance of b with ω. Be sure to write each term as $\omega^e \cdot c$ rather than $c \cdot \omega^e$, and write the terms of the sum in descending order, so that we may use Problem 22.

Problem 24. Calculate $g'(15)(2)$.

Now we will need two more general properties of the ordinals themselves:

Problem 25. Show that a strictly decreasing sequence of ordinals must be finite (i.e.- there is no infinite strictly decreasing sequence of ordinals).

Problem 26. Show that for any m, b, $g'(m)(b)$ is strictly greater than $g'(m)(b+1)$, unless $g(m)(b) = g(m)(b+1) = 0$.

Problem 27. Conclude that any sequence $g(m)(2), g(m)(3), \ldots$ terminates eventually.

4 Challenge: Exponentiation

To prove that the original Goodstein sequences eventually terminate, we need to define exponentiation for more ordinals. If a is an ordinal such that we've defined ω^a, then we can apply the existing rule for natural numbers, and define $\omega^{a+1} = \omega^a \cdot \omega$.

If b is an ordinal that doesn't have an immediate predecessor, then we can't write $b = a + 1$ for any ordinal a, so we're going to have to try something different. The only way to construct b will be through Rule 3, so b must be $\mathrm{lsub}(\{a : a < b\})$. In that case, let $\omega^b = \mathrm{lsub}(\{\omega^a : a < b\})$.

Problem 28. Show that if $a < b$, then $\omega^a < \omega^b$.

Now let's go back to Goodstein sequences. Given a starting number m and $b \geq 2$, define $G'(m)(b)$ by writing $G(m)(b)$ in hereditary base b notation, and replacing every b, in the exponents and otherwise, with an ω.

Problem 29. Show that $G'(m)(b) > G'(m)(b + 1)$ unless they are both 0.

Problem 30. Show that every Goodstein sequence eventually terminates.