1 Examples of Metrics

Last week, we compared the Euclidean metric, which captures the standard geometric notion of distance, with the taxicab metric which captures a different notion. This is not the only way we can redefine the distance between two things, however; more generally we have metrics.

Definition 1 Given a nonempty set $X$, a **metric** on $X$ is a function $d : X \times X \to \mathbb{R}$ such that

i) $d(x, y) \geq 0$ for all $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

$(X, d)$ is called a **metric space**.

Last week we saw two notions of distance in the plane $\mathbb{R}^2$ which we called metrics, the Euclidean and taxicab metrics $d_E$ and $d_T$:

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

**Problem 1** Verify that $d_E$ and $d_T$ are indeed metrics. (Hint: Most of the hard work was done on last week’s worksheet.)

**Problem 2** Define the **discrete metric** on any nonempty set $X$ by

$$d(x, y) = \begin{cases} 
0 & x = y \\
1 & x \neq y
\end{cases}$$

Prove that $d$ is indeed a metric.

**Problem 3** Let $X$ be the set of six-letter words. Define the **Hamming distance** between two words as the number of positions in which they differ. For instance, $d(\text{carrot}, \text{potato}) = 6$ and $d(\text{carrot}, \text{carpet}) = 2$. Verify that the Hamming distance is a metric on $X$. 


Problem 4 For each function $d$ below, determine (with proof) whether or not $d$ is a metric.

a. $X = \mathbb{N}$ (the positive integers), $d(n, m) = \left| \frac{n}{m} - \frac{m}{n} \right|

b. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|

c. $X$ is the set of vertices of a connected graph $G$, $d(x, y)$ is the minimum number of edges of $G$ needed to connect $x$ to $y$ (where we say zero edges are needed to connect a vertex to itself).

d. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \max\{ |x_1 - y_1|, |x_2 - y_2| \}

e. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \min\{ |x_1 - y_1|, |x_2 - y_2| \}

2 Balls in Metric Spaces

As we saw last week, one of the most natural things to consider once we have a distance is what the circles look like. We extend this notion to metric spaces in general.

**Definition 2** Let $(X, d)$ be a metric space. An open ball of radius $r > 0$ centered at $x_0 \in X$ is the set

$$B(x_0, r) := \{ x \in X : d(x_0, x) < r \}$$

It is also possible to talk about closed balls $\overline{B}(x_0, r) := \{ x \in X : d(x_0, x) \leq r \}$, though we will usually stick to open ones.

Problem 5 Draw open balls in $\mathbb{R}^2$ under the Euclidean and taxicab metrics.

Problem 6 Draw or describe the following open balls.

a. $B(0, r)$ (r is a radius $r > 0$) in $X = \mathbb{R}$ with the "standard metric" given by $d(x, y) = |x - y|$.

b. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the metric given by Problem 4d

c. $B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the discrete metric

d. $B((0, 0), 2)$ in $X = \mathbb{R}^2$ with the discrete metric

e. $B(\text{fly}, 2)$ in $X$ the set of 3-letter words with the Hamming distance

f. $B(4, 3)$ in the path graph $P_{10}$ where the vertices are numbered 1, ..., 10 in order (recall the graph theory handout from a few weeks ago) with the metric given by Problem 4c.