General Metrics

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1 Examples of Metrics

Last week, we compared the Euclidean metric, which captures the standard geometric notion of distance, with the taxicab metric which captures a different notion. This is not the only way we can redefine the distance between two things, however; more generally we have **metrics**.

Definition 1 *Given a nonempty set X, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that*

i) $d(x, y) \geq 0$ *for all* $x, y \in X$.

ii) For all $x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$.

iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

*(X, d) is called a **metric space**.*

Last week we saw two notions of distance in the plane $\mathbb{R}^2$ which we called metrics, the Euclidean and taxicab metrics d_E and d_T:

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$
$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Problem 1 *Verify that d_E and d_T are indeed metrics. (Hint: Most of the hard work was done on last week's worksheet.)*

Problem 2 *Define the **discrete metric** on any nonempty set X by*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Prove that d is indeed a metric.

Problem 3 *Let X be the set of six-letter words. Define the **Hamming distance** between two words as the number of positions in which they differ. For instance, $d(carrot, potato) = 6$ and $d(carrot, carpet) = 2$. Verify that the Hamming distance is a metric on X.*

Problem 4 *For each function d below, determine (with proof) whether or not d is a metric.*

a. $X = \mathbb{N}$ *(the positive integers),* $d(n, m) = |\frac{n}{m} - \frac{m}{n}|$

b. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|$

c. *X is the set of vertices of a connected graph G, $d(x, y)$ is the minimum number of edges of G needed to connect x to y (where we say zero edges are needed to connect a vertex to itself).*

d. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

e. $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$

2 Balls in Metric Spaces

As we saw last week, one of the most natural things to consider once we have a distance is what the circles look like. We extend this notion to metric spaces in general.

Definition 2 *Let (X, d) be a metric space. An* ***open ball*** *of radius $r > 0$ centered at $x_0 \in X$ is the set*

$$B(x_0, r) := \{x \in X \ : \ d(x_0, x) < r\}$$

It is also possible to talk about **closed balls** $\overline{B}(x_0, r) := \{x \in X \ : \ d(x_0, x) \leq r\}$, though we will usually stick to open ones.

Problem 5 *Draw open balls in $\mathbb{R}^2$ under the Euclidean and taxicab metrics.*

Problem 6 *Draw or describe the following open balls.*

a. *$B(0, r)$ (r is a radius $r > 0$) in $X = \mathbb{R}$ with the "standard metric" given by $d(x, y) = |x - y|$.*

b. *$B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the metric given by Problem 4d*

c. *$B((0, 0), 1)$ in $X = \mathbb{R}^2$ with the discrete metric*

d. *$B((0, 0), 2)$ in $X = \mathbb{R}^2$ with the discrete metric*

e. *$B(fly, 2)$ in X the set of 3-letter words with the Hamming distance*

f. *$B(4, 3)$ in the path graph P_{10} where the vertices are numbered $1, ..., 10$ in order (recall the graph theory handout from a few weeks ago) with the metric given by Problem 4c.*