

# Week 8: Invariants and Geometric Constructions

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## Problem 1.

Given three segments  $s_1, s_2, s_3$  on the plane, construct a parallelogram with one of the sides equal to  $s_1$  and the diagonals equal to  $s_2$  and  $s_3$ . You may assume such a parallelogram exists.

## Problem 2 (Draw it!).

Consider two rays  $r, \ell$  out of point  $O$ , a segment  $AB$  on  $r$  and point  $C$  on  $\ell$ . Let  $M$  be the midpoint of  $AB$ , let  $D$  be the intersection of  $\ell$  and the line through  $M$  parallel to  $AC$  and let  $E$  be the intersection of  $\ell$  and the line through  $B$  parallel to  $AC$ . Show that  $D$  is the midpoint of  $CE$ .

## Problem 3.

100 chips are arranged in a row. It is allowed to swap two chips located 2 chips apart (that means you can swap any two chips if there is exactly one chip between them). Is it possible to rearrange all the chips in reverse order using that operation multiple times?

## Problem 4.

On a field in the shape of a  $10 \times 10$  grid 9 squares are infested with weeds. A new square can get infested with weeds if at least two of its adjacent squares are infested. Two squares are called adjacent if they share a side. Show that there will always be a square on the field not infested with weeds. *Hint: Consider the perimeter of the shape infested with weeds.*

## Problem 5.

a) Consider an  $n \times m$  table filled with integers. With one operation, you are allowed to take any row or column and negate every number in that row/column. Show that it is possible to make sure every row and column has nonnegative sum using such operations.

b) Same problem with real numbers in the table, not integers.

## Problem 6.

Consider  $n$  segments on the plane with  $2n$  distinct endpoints. The following process is performed: if two segments  $AB$  and  $CD$  intersect, we replace them by segments  $AD$  and  $BC$ . Show that eventually no two segments will intersect.