Ruler and Compass construction of regular polygons

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Polygons

A regular n-gon is a polygon with n sides of equal length and all angles between adjacent sides equal. An alternative description is that all corners of a regular n-gon lie on a fixed circle, and the angle between two adjacent corners, as seen from the center of the circle, is $2\pi/n$. The theme of this work sheet is to construct different regular n-gons.

We list some polygons and comment of the level of conceptual difficulty constructing them with ruler and compass. Note that with increasing n there is also increasing difficulty in the accuracy of the tools needed. There exists free downloadable software (e.g. Geogebra) which simulates ruler and compass constructions without accuracy problems.)

As warm-up we try to construct as many of the polygons on the list as we can. With some experience with basic concepts of ruler and compass constructions, the difficulties up to medium level should be manageable.

1. equilateral triangle ($n = 3$, easy, unless you have never worked with ruler and compass.)
2. hexagon ($n = 6$, almost as easy)
3. square ($n = 4$, medium)
4. regular octogon ($n = 8$, medium)
5. regular 12-gon (medium. Abstract question: suppose you can already do a regular n-gon, can you construct a regular 2n-gon ?)
6. regular pentagon ($n = 5$, difficult)

7. regular 15-gon (medium, provided one has already done 5-gon. Abstract question: suppose you can already construct the regular n -gon and regular m-gon where n and m have no common divisor, can you construct the regular nm-gon?)
8. regular 17-gon (very difficult)
9. Abstract question: suppose $n = 2^r + 1$ is a prime number. Can you construct the regular n-gon? very difficult.)

The algebraic approach

Ruler and compass constructions can be viewed as geometric pendants of algebraic manipulations, a point of view helpful in solving the more difficult cases of constructions of n-gons. We build up the algebraic point of view through a sequence of problems with increasing difficulty.

1. Given two line segments of length a and b , construct a line segment of length $a + b$.

2. Given a number line on which the points $0, x, y$ are marked (assume both are positive) construct the point $x + y$ on the number line.

3. Given a number line on which the points $0, x$ are marked, construct the point $-x$ on the number line.

4. Given a number line on which the points $0, x, y$ are marked (assume x positive and y negative), construct the point $x+y$ on the number line.

5. Given a line segment, construct a line segment of 2 times the length.

6. Given a line segment, construct a line segment of 3 times the length.

7. Given a line segment, construct a line segment of 1/2 times the length.

8. Given a line segment, construct a line segment of 1/3 times the length. (Try this first, but if you have trouble with this, try the exercise ahead.)

9. Given a line segment, construct a line segment of x times the length, where x is the ratio of two other given auxiliary line segments. Hint: this construction uses similar triangles with one common corner.

10. Given a number line on which the points 0, 1, x, and y are marked (assume x, y positive). Construct the point xy on the number line. (Use previous exercise)

11. Given a number line on which the points 0, 1, and p and q are marked. Assume p is positive. Construct the point x on the number line which solves the equation $px+q=0$ (Similar as previous.)

12. Given a number line on which the points 0, 1, and x are marked. Assume x is positive. Construct the point $\sqrt{x}$ on the number line. (Hint this construction uses right triangles, the types for which $a^2+b^2=c^2$ by Pythagoras. The height h above the side c divides the side c into $p+q$. Use $pq=h^2$ or $a^2=pc$, which is half of Pythagoras' theorem. You may want to remember Thales' theorem to construct the right triangle.)

13. Given a number line on which the points 0, 1, p, and p are marked (assume all positive) . Construct two points s and t on the number line such that $s+t=p$ and $st=q$. (Note that s,t are the solutions to the quadratic equation $x^2-px+q=0$, which is the same equation as $(x-s)(x-t)=0$.) Use the hints from the previous exercise.

14. Do the same as before, but assume p positive and q negative.

15. Assume points $0,1,x$ marked on the number line with $x>0$. Construct the point $x^{1/4}$

The above exercises show how to solve arbitrary linear and then quadratic equations geometrically, producing a line segment or a point on the number line representing the solution. Iterating the solution of quadratic equations can solve even more complicated equations such as in the last exercise.

Conversely, every construction with ruler and compass consists of a sequence of basic constructions producing new points by intersecting line with line, or line with circle, or circle with circle. Each of these basic constructions in Cartesian coordinates can be expressed as the solution of a linear (line and line) or quadratic equation (line with circle or cirlce with circle).

The pentagon

The following steps will lead to a construction of the pentagon.

1. Sketch a regular pentagon with all five diagonals. Prove that the three adjacent angles at any given corner of the pentagon are equal.

2. In the same figure consider two adjacent sides of the pentagon and the diagonal that completes the triangle. In the same figure find a similar triangle inside this triangle, and prove rigorously that these triangles are similar isosceles triangles (This means comparing all the angles).

3. Denote the sidelength of the pentagon by 1 and the lengths of each leg of the smaller isosceles triangle in the previous exercise as x. Use both previous exercises to prove $x : 1 = 1 : (1 + x)$. Write a quadratic equation for x. (BTW: This number x is called the golden ratio.)

4. Use the previously discussed geometric solution of quadratic polynomials to construct a line segment of length x.

5. Use this line segment to construct the regular pentagon.

Complex numbers

With 0 and two points x, y marked on the number line we can construct the sum $x + y$ (one dimensional algebra).

Now assume we have three arbitrary points in the plane, not necessarily on one line, marked $0, a, b$ (for emphasis of two-dimensionality we use different letters of the alphabet). There is also a reasonable way to define a sum $a + b$. It is the fourth corner of a parallelogram of which two sides are the line segments $\overline{0a}$ and $\overline{0b}$.

1. Pick three points in the plane, not on a common line. Mark them $0, a, b$ and construct $a + b$.
2. Is this operation commutative?
3. Is this operation associative?
4. How does this construction compare to the usual addition on the real line, i.e. when the points $0, a, b$ happen to be on a single line?
5. Given two points marked 0 and a in the plane, which point in the plane is $-a$?

Likewise, if we are given four points on a line marked $0, 1, x, y$ (we always assume the points marked 0 and 1 are different), then we can contstruct the point xy. Now given any four points in the plane marked $0, 1, a, b$, there is also a reasonable way to construct a product ab described as follows:

1) if $|a|$ denotes the distance from a to 0, then we have the ratio

$$|ab| : |b| = |a| : |1|$$

(that is unless $|a| = 0$, in which case $ab = 0$) and 2) the angle $(1, 0, ab)$ is the sum of the angles $(1, 0, a)$ and $(1, 0, b)$.

1. Pick four points in the plane, (for full appreciation no three should be on on a common line). Mark them $0, 1, a, b$ and assume $0 \neq 1$. Construct ab.
2. Is this operation commutative?

3. Is this operation associative?

4. Do we have the distributive law $(a+b)c = ac + bc$ (this is easiest when we interpret multiplication by c as a rotation by the angle $(1, 0, c)$ combined with a dilation by the factor $|c|/|1|$.

5. Given three different points marked $0, 1$ and a, which point in the plane is $1/a$?.

The operations of addition and multiplication above give the plane a very similar algebraic structure as the number line. Points on the line are called real numbers, while the points in the plane are called complex numbers. The real numbers are embedded into the complex numbers, they lie on the line passing through the points 0 and 1.

Introducing a Cartesian coordinate system one can describe complex numbers via a pair of real numbers: $a = (x, y)$. Typically one arranges things such that the complex 0 has coordinates $(0, 0)$ and the complex 1 has coordinates $(1, 0)$. The point $(0, 1)$ is usually denoted by i and called the imaginary unit. Note that with two points marked $0, 1$, there is a choice of two coordinate systems as above, which differ by a reflection across the real line.

1. Show that $i^2 = -1$ using the definition of the product above.

2. For any complex number a in the plane construct two points s, t with $s^2 = t^2 = x$.

3. Solve an arbitrary quadratic equation in complex numbers. (This is a bit more elaborate exercise, that can be skipped at first reading. That every quadratic equation has two solutions is a distnguishing feature between real and complex numbers, note that the equation $x^2 = -1$ cannot have a real solution.)

4. How can we express the sum and product of complex numbers in coordinates?

The heptadecigon ($n = 17$)

Sketch a regular seventeen-gon in the complex number plane with center at 0 and one corner at 1. Let a denote one of the corners adjacent to 1. Note: using actual physical ruler and compass, it requires great care in accuracy for the result of the following steps to be satisfactory.

1. Find the powers a^2, a^3 etc. in your figure. Prove $a^{17} = 1$.

2. Prove that $\sum_{n=1}^{17} a^n = 0$. (Hint: denote the left hand side by b and prove first $ba = b$.

3. Note that $a^{-n} = a^{17-n}$. We use this to write a^{-1} for a^{16} etc. Identify
$$\sum_{n=1}^{8} a^n + a^{-n}$$
by comparing with the previous exercise.

4. Denote $s = a^{-8} + a^{-4} + a^{-2} + a^{-1} + a^1 + a^2 + a^4 + a^8$ and $t = a^{-7}+a^{-6}+a^{-5}+a^{-3}+a^3+a^5+a^6+a^7$. Identify s and t approximetely in your figure. Prove they are on the real number line (hint: pair a^n with a^{-n}). Which is positive and which is negative?

5. Prove $s+t = -1$ and $st = -4$. The latter is the most tedious calculation of this worksheet, maybe one can do this smartly/pictorially? Construct s and t by solving geometrically a quadratic equation (involving only real numbers).

6. Denote $u = a^{-4} + a^{-1} + a^1 + a^4$ and $v = a^{-8} + a^{-2} + a^2 + a^8$. Prove these are real numbers and approximately identify them in the figure.

7. Prove $u + v = s$ and $uv = -1$. Use this to construct u and v.

8. Denote $p = a^{-7} + a^{-6} + a^6 + a^7$ and $q = a^{-5} + a^{-3} + a^3 + a^5$. Determine p and q similarly to above.

9. Denote $c = a^{-1} + a^1$ and $d = a^{-4} + a^4$, identify these in the figure, determine $c + d$ and cd and solve geometrically for c and d.

10. Using c, construct a and thus the regular 17-gon. Note that this requires solving a quadratic equation in complex numbers. However, the solution is simple when studying the intersection point of the real line and the line connecting the points x and x^{-1}.

The above construction seemed to depend on a series of algebraic miracles. To shed some light on these miracles, one should note that $-8, -4, -2, -1, 1, 2, 4, 8$ are exactly the remainders of square numbers after division by 17. The numbers $-4, -1, 1, 4$ are exactly the remainders of fourth powers after division by 17. The numbers -1,1 are exactly remainders of eighth powers after division by 17. With this idea and an abstract approach the above miracles gain a lot of clarity. Indeed one can repeat the scheme whenever a number $n = 2^m + 1$ is a prime number ("Fermat number") such as for example $2^4 + 1 = 17$. Then one can similarly construct these n-gons. The 5-gon is also of this type (even the 3-gon) though the construction of the pentagon was discovered "by bare hands" and described in Euclids Elements around 300 BC, long before the discovery of the above scheme. The next Fermat number is 257, hence one can construct the 257-gon.

We develop the construction of the pentagon using the same scheme as above.

1. identify the pentagon as having corners $1, a, a^2, a^3, a^4$ for some complex number solving $a^5 = 1$.
2. Set $x = a + a^{-1}$ and $t = a^2 + a^{-2}$. Identify these (real numbers) in your figure and verify $s + t = -1$, $st = -1$
3. Use s and t to construct the pentagon.

The above scheme was developed by Carl Friedrich Gauss in the late 17 hundreds. Understanding the algebraic structure behind the scheme very carefully (Galois theory), it could even be proved that there are n-gons that cannot be constructed by ruler and compass alone (Gauss-Wantzel theorem). The first example is the 7-gon. Indeed, the only n-gons that can be constructed by ruler and compass are the ones that can be done by the methods developed on this worksheet.