Winter Quarter Game 1

Nikita

1 Algebra

1. The father was asked how old his two sons were. The father replied that if you add the sum of these ages to the product of their ages, you get 34. How old are his sons?

2. Simplify:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\dots\left(1-\frac{1}{225}\right)$$

3. Represent the number $2 \cdot 2020^2 + 2 \cdot 2021^2$ as the sum of the squares of two natural numbers.

4. What happens if $2^{62} + 1$ is divided by $2^{31} + 2^{16} + 1$?

5. Factor the polynomial $x^8 + x^4 + 1$ into 4 polynomial factors (polynomials can have non-integer coefficients).

2 Geometry

1. What is the largest possible distance between two points located on the boundary of a 3×4 rectangle?

2. From point A outside of a circle one draws a tangent line and a secant line. Distance from point A to the tangency point is 16, and the distance from point A to one of the intersection points of the secant with the circle is 32. Find the radius of the circle if the distance from its center to the secant line is 5.

3. In triangle ABC, points P is taken on the base AC. Straight line BP intersects the median AM at point Q. It is known that $AQ = 6$, $AM = 9$ and $AP = 5$. Find AC.

4. Let P lie on the quadrilateral $ABCD$ on edge AB. Angles BAD, ABC and CPD are $90°$, $AB = 40$, $BC = 27$, $AD = 13$. In what relation does the point P divide AB?

5. AB is a circle diameter, BC and CDA are a tangent and a secant. Find the ratio $CD : DA$ if BC is equal to the radius of the circle.

3 Combinatorics

1. How many ten-digit numbers have digits that sum to 4?
2. How many solutions does the equation $x + y + z = 20$ have such that x, y, and z are all positive integers?
3. A regular 9-gon is given. How many ways can you choose three of its vertices such that the triangle between those vertices is an isosceles triangle?
4. You have an unpainted 8×8 chessboard. How many ways can you color the board's squares black and white so that there are 31 black squares in total and no two black squares share a common side?
5. The game of Preferans is played by three players using a special French-suited 32-card deck. At the beginning, each player receives 10 cards, and the remaining 2 cards are set aside. How many different initial dealings of cards are possible?

4 Number theory

1. Find the smallest natural n for which $(n+1)(n+2)(n+3)(n+4)$ is divisible by 1000.

2. Find all rectangles with natural number length sides whose perimeter is equal to the area.

3. Mary multiplied several natural numbers and got 224, with the smallest number being exactly half the largest. How many numbers did Mary multiply?

4. How many natural numbers n, less than 10000, for which $2^n - n^2$ is divisible by 7?

5. How many (unordered) pairs of natural numbers are there with the least common multiple of 2000?

5 Text problems

1. Cockroach Valentin announced that he can run at a speed of $50\frac{m}{min}$.

   People did not believe him, and rightly so: in fact, Valentin confused everything and thought that in a meter it was 60 cm, and in a minute – 100 seconds. At what speed (in "normal" $\frac{m}{min}$) does the Valentin run?

2. A carpenter sawed an 8×8 chessboard into squares in 70 minutes. How long will it take for him to saw the same board into 2×2 squares?

3. The cells of a 3×3 square were filled with the numbers from 1 to 9 and the sum of the numbers in each row and in each column was calculated. What is the largest possible number of consecutive integers among these sums?

4. Olivia and Emma each have n coins. Each coin has a denomination of 1 cent, 50 cents or 1 dollar. It turns out that the girls have the same amount of money, but the sets of coins did not match. What is the smallest possible value of n?

5. There are 16 teams participating in the California volleyball tournament. Every two teams play against each other **twice**. The teams that take the first 8 places are going to the interstate tournament. The teams are ordered by the number of wins; in case of an equal number of victories for several teams, these teams are ordered by lot. There are no draws in volleyball. What is the smallest number of victories needed to guarantee the pass to the interstate tournament no matter what?