

# ORMC: MIXED PROBLEM SESSION

OLYMPIAD GROUP 1, WEEK 6

**Problem 1.** Define a sequence  $x$  recursively by the initial value  $x_0 = 2021$  and  $x_{n+1} = x_n^2 - 1$ . Show that  $x_{2n}$  is divisible by  $2021^{2^n}$ .

**Problem 2.**

(a) Let  $a > b > 0$  be real numbers. Show that

$$\frac{a+1}{b+1} < \frac{a}{b}.$$

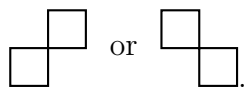
(b) Let  $a > b > 0$  be *integers* such that

$$a \mid b, \quad a+1 \mid b+1, \quad \dots, \quad a+n \mid b+n.$$

Show that  $n < \frac{a}{b}$ .

**Problem 3.** Let  $\triangle ABC$  be an acute triangle. Let  $M, N$  be the midpoints of the sides  $AB$  and  $AC$  respectively, and  $X$  be the midpoint of the segment  $MN$ . Construct points  $U, V$  on  $BC$  such that  $XU \parallel AB$  and  $XV \parallel AC$ . Show that  $AX, MV$  and  $NU$  are concurrent lines.

**Problem 4.** We have a  $10 \times 10$  board. In how many ways can we pair up the 100 cells into 50 pairs, so that each pair consists of cells sharing exactly one corner? i.e.



**Problem 5.** Let  $n$  be a positive integer, and assume that  $p = 2^n + 1$  is a prime.

(a) Show that  $n$  is a power of 2. *In this context,  $p$  is called a Fermat prime.*

(b) Find all residue classes  $\hat{a}$  modulo  $p$  such that  $\hat{a}^6 = \hat{1}$ . (*This isn't related to part (a).*)

**Problem \*6.** Consider a  $2021 \times 2021$  board, where on line  $a \in \{1, 2, \dots, 2021\}$  and column  $b \in \{1, 2, \dots, 2021\}$  we write the number  $a \times b$ . Two cells are called neighbors if they share a common side or corner; so two neighboring cells can be of type  $\square\square$ ,  $\square$ ,  $\square$  or  $\square$ .

A pair of neighboring cells (in this board) is called *perfect* if the two cells contain consecutive perfect squares (e.g., 16 and 25).

(a) Show that a pair of  $\square\square$  or  $\square$ -neighboring cells cannot be perfect.

(b) Show that if a pair of  $\square$ -neighboring cells is perfect, then it lies on the main diagonal.

(c) Is there any perfect pair of  $\square$ -neighboring cells?