The Taxicab Metric

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1 The Taxicab Metric in the Plane

Normally when we discuss length in geometry between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in the plane, we are really talking about the **Euclidean distance** (or Euclidean *metric*) on $\mathbb{R}^2$

$$d_E(x, y) = d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

which measures the straight-line distance between them. This notion of distance is not as meaningful to, for instance, a taxicab driver in New York City. He or she cannot drive through a building to arrive at the destination in a straight line, and is more likely to say that a location is "four blocks away". To capture this notion of distance, we define a new *metric*, or distance function.

Definition 1 *The **taxicab distance** between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in the real plane $\mathbb{R}^2$ is defined by*

$$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

Problem 1 *For each of the following pairs of points $x, y \in \mathbb{R}^2$, compute their Euclidean distance $d_E(x, y)$ as well as their taxicab distance $d_T(x, y)$. Which one is larger?*

a. $x = (0, 0)$, $y = (1, 0)$

b. $x = (0, 4)$, $y = (3, 0)$

c. $x = (-1, 6)$, $y = (4, -6)$

d. $x = (2, 3)$, $y = (26, 10)$

Solution: Students should notice that the taxicab distance is always as large as the Euclidean distance

a. $d_E(x, y) = \sqrt{(1-0)^2 + (0-0)^2} = 1$, and $d_T(x, y) = |1 - 0| + |0 - 0| = 1$.

b. $d_E(x, y) = \sqrt{(4-0)^2 + (0-3)^2} = 5$, and $d_T(x, y) = |4 - 0| + |0 - 3| = 7$.

c. $d_E(x, y) = \sqrt{(4+1)^2 + (-6-6)^2} = 13$, and $d_T(x, y) = |4 + 1| + |-6 - 6| = 17$.

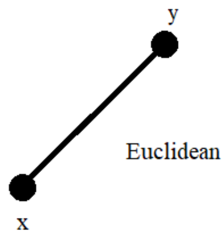
d. $d_E(x, y) = \sqrt{(26-2)^2 + (10-3)^2} = 25$, and $d_T(x, y) = |26 - 2| + |10 - 3| = 31$.

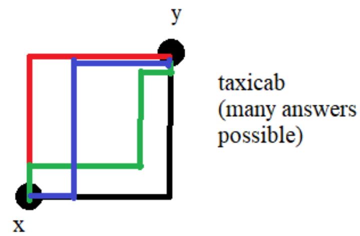
Problem 2 *Pick any x, y in the plane $\mathbb{R}^2$.*

a. Draw a path from x to y that corresponds to the distance $d_E(x, y)$, in the sense that its length equals $d_E(x, y)$.

b. Draw a path from x to y that corresponds to the distance $d_T(x, y)$, in the sense that its length equals $d_T(x, y)$. Can you draw more than one such path?

Solution: Assuming x, y do not share the same first or second coordinate, the drawings should look like:





In Euclidean geometry we have various classical shapes, such as the circle, which can be considered as the set of all points the same distance from the center, and the line segment between two points p, q, which can be considered as the set of all points x such that $d_E(p, x) + d_E(x, q) = d_E(p, q)$. It is natural to ask what these same sets look like if we use the taxicab distance instead of the Euclidean one.

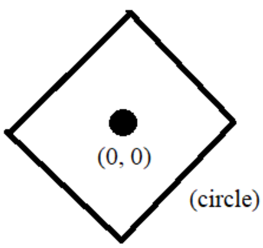
Problem 3 *a. Draw the* ***taxicab unit circle***

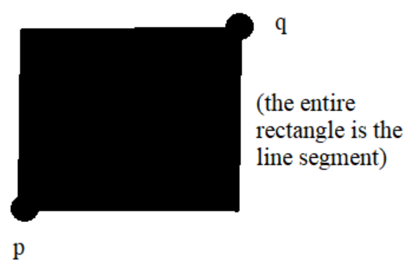
$$\{x \in \mathbb{R}^2 \ : \ d_T(x, (0, 0)) = 1\}$$

b. Draw the ***taxicab line segment*** *between p and q*

$$\{x \in \mathbb{R}^2 \ : \ d_T(p, x) + d_T(x, q) = d_T(p, q)\}$$

Solution: Again assuming p, q do not share the same first or second coordinate,





2 Properties of the Taxicab Metric

We will now prove some useful properties of both the Euclidean and taxicab metrics; they turn out to not be so different after all.

Problem 4 *a. Prove that for any $x, y \in \mathbb{R}^2$, $d_E(x, y) \geq 0$ and $d_T(x, y) \geq 0$.*

b. Prove that we have equality in part (a) in either case if and only if $x = y$.

c. Prove that for any $x, y \in \mathbb{R}^2$, $d_E(x, y) = d_E(y, x)$ and $d_T(x, y) = d_T(y, x)$.

Solution:

a. The square root of a nonnegative real number is a nonnegative real number, and absolute values are nonnegative.

b. In both cases, everything is nonnegative so they can only be zero if $x_1 - y_1 = x_2 - y_2 = 0$, in which case $x = y$.

c. Use the fact that $(a - b)^2 = (b - a)^2$ and $|a - b| = |b - a|$ for any two real numbers a, b.

Recall from Euclidean geometry that we have the **triangle inequality**

$$AC \leq AB + BC$$

for any three points A, B, C in the plane. (where the above quantities denote the lengths of the respective Euclidean line segments)

Problem 5 *a. Using Problem 2a, explain why we have the **Euclidean triangle inequality** in the plane*

$$d_E(x, z) \leq d_E(x, y) + d_E(y, z)$$

b. (i) Prove that for any real numbers a, b

$$|a + b| \leq |a| + |b|$$

*(ii) Prove the **taxicab triangle inequality** in the plane*

$$d_T(x, z) \leq d_T(x, y) + d_T(y, z)$$

(Hint: Expand out each of $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$, and write down both sides of the above expression in terms of these. How can you use part (i)?)

c. Using Problem 2 and part (a), explain why for any $x, y \in \mathbb{R}^2$

$$d_E(x, y) \leq d_T(x, y)$$

When do we have equality?

Solution:

a. Letting $A = x$, $B = y$, $C = z$, by Problem 2a the lengths of the line segments are exactly the distances between the points.

b. (i) There are four cases to check.

$$|a + b| = a + b = |a| + |b| \text{ if } a, b \geq 0$$
$$|a + b| = -a - b = |a| + |b| \text{ if } a, b \leq 0$$
$$|a + b| = |a - (-b)| \leq a + (-b) = |a| + |b| \text{ if } a \geq 0, b \leq 0$$
$$|a + b| = |b - (-a)| \leq b + (-a) = |a| + |b| \text{ if } b \geq 0, a \leq 0$$

(ii) Using the previous part,

$$d_T(x, z) = |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| = d_T(x, y) + d_T(y, z)$$

c. Take a taxicab and Euclidean path from x to y. By the Euclidean triangle inequality, the Euclidean path will always be shorter, with equality if and only if x and y only differ in one of their coordinates.

3 Distance in Higher Dimensions

While the analogy of taxicabs breaks down in other dimensions, we can still discuss both the Euclidean and taxicab metrics in $\mathbb{R}^n$ as well:

$$d_E((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$
$$d_T((x_1, ..., x_n), (y_1, ..., y_n)) = |x_1 - y_1| + ... + |x_n - y_n|$$

Problem 6 *For each of the following ordered n-tuples of points $x, y \in \mathbb{R}^n$, compute their Euclidean distance $d_E(x, y)$ as well as their taxicab distance $d_T(x, y)$.*

a. $x = (1, 1, 3, 4)$, $y = (-1, 7, 1, 0)$

b. $x = (0, 1, 2, 3, 4)$, $y = (5, 6, 7, 8, 9)$

Solution:

a. $d_E(x, y) = \sqrt{(1+1)^2 + (1-7)^2 + (3-1)^2 + (0-4)^2} = \sqrt{60}$
and $d_T(x, y) = |1+1| + |1-7| + |3-1| + |0-4| = 14.$

b. $d_E(x, y) = \sqrt{(5-0)^2 + (6-1)^2 + (7-2)^2 + (8-3)^2 + (9-4)^2} = 5\sqrt{5}$
and $d_T(x, y) = |5-0| + |6-1| + |7-2| + |8-3| + |9-4| = 25.$

We would very much like the properties we proved in Problems 4 and 5 to remain true in higher dimensions; indeed they do.

Problem 7 *a. Prove that for any $x, y \in \mathbb{R}^n$, $d_E(x, y) \geq 0$ and $d_T(x, y) \geq 0$.*

b. Prove that we have equality in part (a) in either case if and only if $x = y$.

c. Prove that for any $x, y \in \mathbb{R}^n$, $d_E(x, y) = d_E(y, x)$ and $d_T(x, y) = d_T(y, x)$.

Solution: The solution is the exact same reasoning as Problem 4.

Problem 8 *Prove the taxicab triangle inequality in $\mathbb{R}^n$ (the Euclidean triangle inequality is also true, but well beyond the scope of this worksheet)*

$$d_T(x, z) \leq d_T(x, y) + d_T(y, z)$$

Solution: Similarly to Problem 5b, we expand out each coordinate and use $|a + b| < |a| + |b|$.

$$\begin{aligned} d_T(x, z) &= |x_1 - z_1| + ... + |x_n - z_n| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + ... + |x_n - y_n| + |y_n - z_n| = d_T(x, y) + d_T(y, z) \end{aligned}$$

4 Bonus Section: More Taxicab Geometry

Problem 9 *Consider the following shapes in $\mathbb{R}^2$, which are based on typical Euclidean shapes.*

a. Draw the taxicab ellipse with foci $p, q \in \mathbb{R}^2$ and parameter $b > d_T(p, q)$

$$\{x \in \mathbb{R}^2 \,:\, d_T(x, p) + d_T(x, q) = b\}$$

b. Draw the taxicab perpendicular bisector of $p, q \in \mathbb{R}^2$

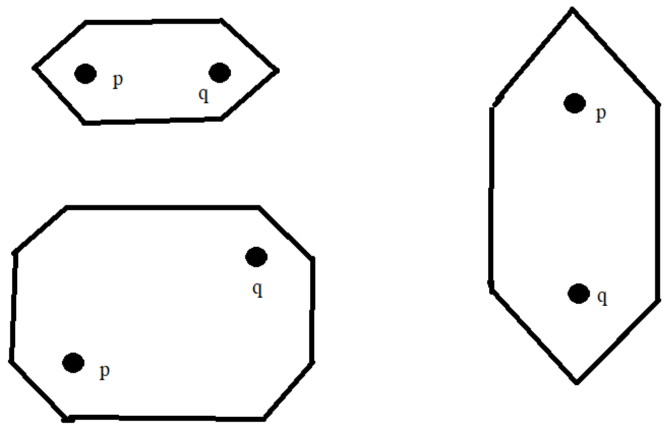
$$\{x \in \mathbb{R}^2 \,:\, d_T(x, p) = d_T(x, q)\}$$

c. Draw the taxicab hyperbola with foci $p, q \in \mathbb{R}^2$ and parameter $b \in \mathbb{R}$

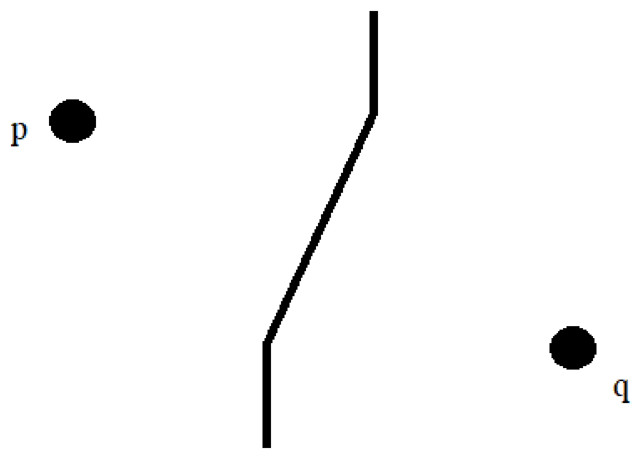
$$\{x \in \mathbb{R}^2 \,:\, |d_T(x, p) - d_T(x, q)| = b\}$$

Solution:

a. Several cases (including p, q being directly horizontal/vertical and otherwise)



b. Students' drawing may differ if p and q are chosen at a different angle.



c. Students' drawings may differ if points and/or b are chosen differently.