Exponential Functions Part I

UCLA Olga Radko Math Circle Beginners 2

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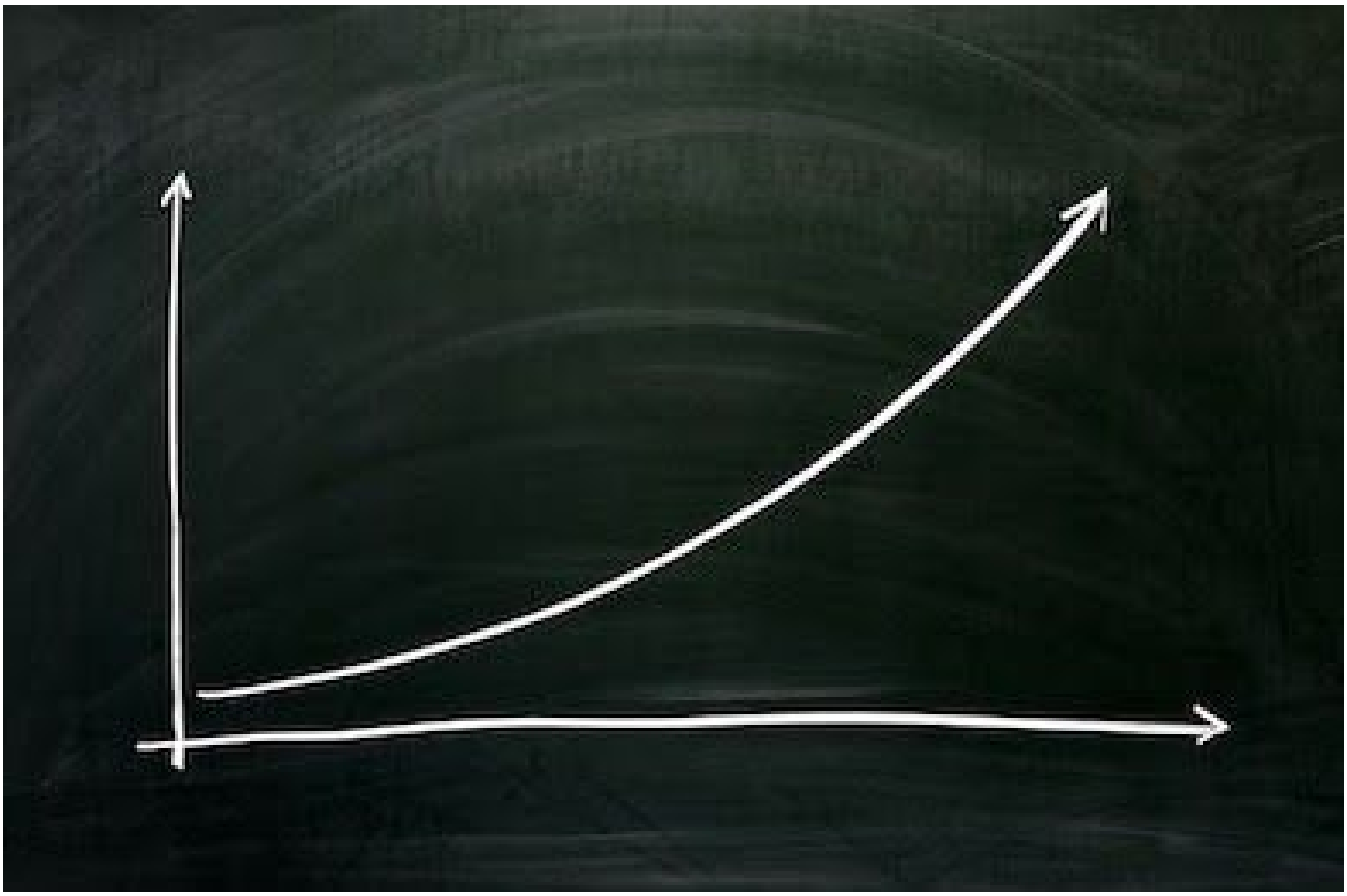

We previously studied Egyptian Math, where we discussed powers of 2 and 10. Today, we will dive deeper into the world of powers and exponents.

Warm-Up

Alice, Bob, Cole, and Donna were picking mushrooms. Alice picked more mushrooms than other children. There was a child who picked fewer mushrooms than Donna.

a. *Is it true that the girls picked more mushrooms than the boys? Circle the correct answer.*

True ***False***

b. *Explain your choice.*

A > B > D > C. This means that A + D > B + C. Since, A > B > C and D > C, so A+D would be bigger. If A picks 4, B picks 3, D picks 2, and C picks 1, then girls would have picked 6, where boys would have picked 3.

(or A > C > D > B, so A+D > B+C

Exponential Functions

Problem 1: Most of the time, exponents are just a shorthand for multiplication. See if you can write out the following examples using *multiplication* instead of *exponents*.

a. What is the value of 2^4?

   i. $2^4 = 2 \times 2 \times 2 \times 2 = 16$

b. What about the value of 3^5?

   i. $3^5 =$ 3x3x3x3x3 $= 243$

c. What about 4^4?

   i. $4^4 =$ 4x4x4x4 $= 256$

d. Let's try a challenging one. 5^{2021}?

   i. $5^{2021} = \overbrace{5 \times \ldots \times 5}^{\text{2021 times}}$, where 5 is multiplied to itself 2021 times. (The goal of this question is to familiarize the students with the […] notation and what it means).

ii. We know that the value of 5^{2021} is going to be a ***really*** big number. So, let's try to use a calculator to find its value. What happens when we use a calculator to find its value?

Overflow Error (The value was so big that the calculator wasn't able to store it correctly)

iii. How can exponents help us with this problem? (Why might exponents be useful in writing down really big numbers?)

Exponents provide us with a neat way of writing down really big numbers. We know that the answer is going to be a number and its exponent form is useful in representing in a concise manner.

The Base

Problem 2: Now, let's try to generalize this. What might b^{2021} mean? (***b*** is a **natural number**).

a. So far, we have been considering **natural numbers**. What is a **natural number**?

Positive whole integers/numbers.

b. Is 0 a **natural number**? Explain.

0 is not a natural number because 0 was discovered much later the counting numbers were discovered. Also, zero does not have a positive or negative value and natural numbers are whole positive integers.

i. If we defined ***b*** to be a **natural number**, based on your previous answer, what can you say about ***b***?

$b > 0$

c. What would happen if we let ***b*** = 1?

i. $b^2 = (1)^2 = 1$

ii. $b^6 = (1)^6 =$ 1 _______________

iii. $b^{2020} = (1)^{2020} =$ 1

iv. What happens when we let $b = 1$?

The answer is simply 1 since 1 multiplied to itself is 1.

d. Now, let's consider negative 1 ($b = -1$).

i. $b^2 = (-1)^2 =$ 1

ii. $b^3 = (-1)^3 =$ -1

iii. $b^4 = (-1)^4 =$ 1

iv. $b^5 = (-1)^5 =$ -1

v. $b^{12} = (-1)^{12} =$ 1

vi. $b^{13} = (-1)^{13} =$ -1

vii. Do you see a pattern? If so, what do you notice?

If the exponent is even, answer is 1. If the exponent is odd, answer is -1.

viii. What happens when **b** is negative? *(Try doing the same problems but when **b = 1**)*

It simply causes the negative sign to oscillate between answers.

e. The ***b*** we have explored so far (the *factor* being multiplied by itself) is called a **base.** Based on your answers from a-d, what kind of numbers can we use as the **base** for an exponential function?

b > 0; b does not equal 1 since this case is trivial.

f. Based on everything we have learned so far, what might b^{2021} mean?

b x … x b (make sure to put a bracket underneath and write 2021 to indicate that b is being multiplied to itself 2021 times)

Red Chilli Pepper Problem

A few sparrows decided to land on a line of trees along a street. Trying to land one bird on a tree, they found out that there were not enough trees for four sparrows. They landed two birds on one tree instead and one tree remained unoccupied. How many sparrows and how many trees were there?

Teacher's notes

Problem 24.15: there are two observations that help to solve the problem.

- There were four more sparrows than trees. Therefore, the number of the sparrows was five or more.

- The number of the sparrows was even. Indeed, they all landed two birds on a tree. So, the minimal number of the sparrows to check is 6.

From here, one can proceed by trial and error.

# of sparrows	6	8	10	12	14
# of trees	2	4	6	8	10

10 sparrows and 6 trees is the first pair of numbers that works. For 12, 14, and more sparrows, there is more than one tree left vacant, so 10 sparrows and 6 trees is the only solution.

The Exponent/Power

<u>Problem 3:</u> The **exponent/power** is the small number on the top right of our exponential functions. Based on the problems we have done so far, what does the **power** tell us?

It tells us how many times we multiply the base to itself.

As we continue with our exploration of powers, you'll see that the exponential function has two very important properties. Let's take a look at the first one through some examples. (Write your answer in the exponential form)

Problem 4:

a. $4^2 \times 4^2 = 4 \times 4 \times 4 \times 4 = 4^4$

b. $2^3 \times 2^2 =$ ____ $= 2^5$

c. $3^4 \times 3^5 =$ ____ $= 3^9$

d. $6^{10} \times 6^{20} =$ ____ $= 6^{30}$

e. $5^{2021} \times 5^{100} =$ ____ $= 5^{2121}$

f. What do you notice about what we have done so far?

   We seem to be adding the exponents for the problems above.

Problem 5: Let's see how we can generalize what we found. Suppose we have two exponents ***n*** and ***m***. Again, we are considering natural numbers for our exponents.

a. $2^n \times 2^m =$ ______________________ $= 2^{n+m}$

b. We've generalized our **base** to be represented as **b.** So what does $b^n \times b^m$ *equal*?

$b^n \times b^m =$ ______________________ $= b^{n+m}$

Thus, we have found one of the properties of an exponential function. Let's take a look at another property

Problem 6: *Try these problems. (Make sure to write the answer in the exponential form)*

a. $(2^2)^2 = 2^2 \times 2^2 = 2^4$

b. $(3^4)^2 = 3^4 \times 3^4 = 3\times3\times3\times3\times3\times3\times3\times3 = 3^8$

c. $(4^3)^4 = 4^3\times4^3\times4^3\times4^3 = 4^{12}$

d. $(5^{2021})^2 = 5^{2021}\times5^{2021} = 5^{4042}$

e. What do you notice about what we have done so far?

We seem to be multiplying the exponents.

Problem 7: Let's see how we can generalize what we found. Suppose we have two exponents ***n*** and ***m***. Again, we are considering natural numbers for our exponents.

c. $(2^n)^m = \overbrace{2^n\times2^n\times\cdots\times2^n}^{m \text{ times}} = 2^{n\times m}$

d. We've generalized our **base** to be represented as **b.** So what does the following *equal*?

$$(b^n)^m = \overbrace{b^n\times b^n\times\cdots\times b^n}^{m \text{ times}} = 2^{n\times m}$$

Thus, we have found another property of an exponential function. These two properties are very important because as we continue to learn more about exponential functions, these two properties **must** hold true.

Comparing Numbers

Problem 8: Using everything we have learned so far to circle the larger value.

a. *What if they have the **same base**?*
   i. 2^5 vs 2^8

   ii. 3^7 vs 3^{10}

   iii. 4^{15} vs 4^8

   iv. In order for 2^m to be larger than 2^n, what must be true of **m** and **n**?

   $m > n$

b. *What if they have a **different base**? Is there something we can do to make them have the **same base**?*

   i. 8^2 vs 2^4 => $(2^3)^2$ vs 2^4 => 2^6 vs 2^4 => 8^2

   ii. 4^3 vs 16^9 => $(4^2)^9$ vs 4^3 => 4^{18} vs 4^3 => 16^9

   iii. 125^2 vs 5^3 => $(5^3)^2$ vs 5^3 => 5^6 vs 5^3 => 5^6

   iv. 81^4 vs 3^5 => $(3^4)^4$ vs 3^5 => 3^{16} vs 3^5 => 3^{16}

c. *What if they have the **same exponent?***

   i. 3^5 vs 2^5

   ii. 3^{2021} vs 5^{2021}

   iii. $3^5 \times 5^5$ vs $2^5 \times 8^5$

   iv. $9^6 \times 2^6$ vs $3^6 \times 7^6$

v. Suppose we have two different bases, **b** and **h** with the properties we have defined so far. In order for $\mathbf{b^9}$ to be larger than $\mathbf{h^9}$, what must be true of **b** and **h**?

$b > h$

*d. (Challenge) What if they have **different exponents**? Is there something we can do to make them have the same **same exponent**?*

i. 8^3 vs 5^9 => 8^3 vs $(5^3)^3$ => $5^3 > 8$ => 5^9

ii. 16^2 vs 3^8 => 16^2 vs $(3^4)^2$ => $3^4 > 16$ => 3^8 OR
=> 3^8 vs $(2^4)^2$ => $3 > 2$ => 3^8

Something to think about: So far, we've looked at exponents that are natural numbers (1, 2, …). But what would happen if it was 0? or a negative number? or even a fraction?

Challenge Problems

1. (Today is Sunday). I have a cold. I came to school anyway and caused two friends to get sick. The next day, my two friends came to school and each infected two other friends. **If this pattern continues, how many kids will get sick on Friday?** *Make sure to write your answer in exponential form.*

Sun: 2 ; Mon: 4; ….. => on day x, we infect 2^x => 2^5

a. What about the next Friday?

2^{12}

b. The one after that?

2^{19}

2. What is 2^{2^3} ?

=> 2^8 => 256

3. Cassandra and Paul are playing a game, where Paul tries to figure out which of the following would be the biggest. Cassandra gives Paul one hint: *"**x** is some number <u>greater than zero</u> but <u>less than one.</u>"* Which one should Paul pick?

x^2 x^3

x^4 $-x$

4. If an asteroid traveling at a rate of $2 * 10^3$ miles per day is currently $8 * 10^{15}$ miles away from Earth, how many days would it take to reach earth? (distance = rate * time)

(8 x 10 … 10) / (2 x 10 x 10 x 10) => Cancel out three 10's from above and reduce 8/2 to 4. => 4 x 10^{12} days.

5. The speed of light is $3 * 10^8$ meters per second. If a single photon of light travels for 500 minutes, how far does it travel? (There are 60 seconds in a minute and distance traveled = speed * time traveled)

500 x 60 = 30000 seconds = 3 x 10 x 10 x 10 x 10 = 3 x 10^4 seconds.

$3 \times 10^8 \times (3 \times 10^4) = 9 \times 10^{12}$ meters