DRAWING SECTIONS IN THREE DIMENSIONS

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1. Projections of three-dimensional figures

In a high school geometry course, you have probably run into both two-dimensional and three-dimensional geometry problems. Although we live in three-dimensional space and routinely manipulate three-dimensional objects, we find objects in two-dimensional geometry easier to imagine. The precise reason for this is probably psychological in nature; perhaps it is the consequence of our essentially two-dimensional visual system, or perhaps it is because we grow up learning to draw on flat sheets of paper. Regardless of the reason, we come accross the fact that we do not have any practical ways of drawing three-dimensional pictures; we can only draw a two-dimensional *projection* of a 3D object and then rely on our imagination to picture that object in three dimensions.

Let us first examine mathematically what we mean by a projection of a three-dimensional figure. To do so, let us fix a plane Π (the plane of the projection) and some direction ℓ which is not parallel to Π. If P is some point in three-dimensional space, it projection onto Π will be denoted by $\bar{P}$. To find $\bar{P}$ we first draw a line ℓ' through P and parallel to ℓ. The point $\bar{P}$ is then the intersection of ℓ' and Π (see Figure 1.1).

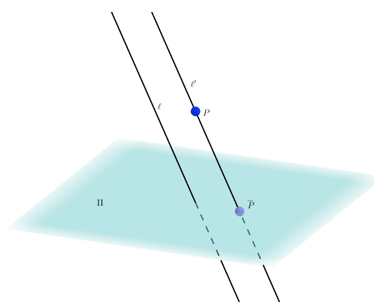


Figure 1.1. Projection $\bar{P}$ of a point P along the line ℓ onto the plane Π.

An example of this process is seen in everyday life when we look at shadows cast by objects. Figure 1.2 shows the shadow cast by a chair. In this figure, the plane of projection Π is the plane of the pavement, while the direction of the rays of the sun specify the direction of projection ℓ. The shadow is the projection of the chair onto the plane Π (see Figure 1.2).

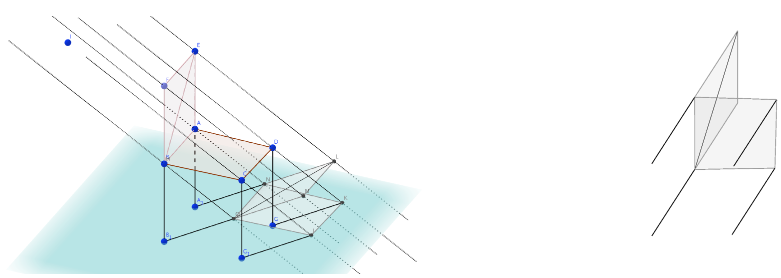


Figure 1.2. The shadow cast by a chair and the projection of a chair onto the plane of the floor.

The resulting projection is not a faithful image of three-dimensional space. Each point Q on the projection plane Π is the projection of many points (in fact, of the entire line, the parallel to the line of projection ℓ passing through Q). Fortunately, there are a few simple observations that can be made:

Fact 1. *The following hold:*

(1) *The projection of a point is a point. However, several distinct points may have the same projection.*
(2) *The projection of a line is either a line or a point. The projections of two parallel lines either coinside or are parallel.*
(3) *The projection of a plane is either the whole plane of projection* Π*, or a line.*
(4) *If* π *is a plane which is not parallel to the line of projection* ℓ*, then distinct points on* π *have distict projections.*

Exercise 2. Draw pictures of the following phenomena:

(1) Two distinct points having the same projection.
(2) A line whose projection is a point and a line whose projection is a line.
(3) A plane whose projection is a line and a plane whose projection is Π.

We will always denote by Π the plane of projection (i.e., our piece of paper).

Because of Fact 1.1 above, it may happen that two non-intersecting lines intersect at a point. (Give an example; such lines are called *skew*). Because of this, it is difficult to tell from a 2D projection whether any two given lines in 3D intersect or not. Typically, projections of 3D objects have "**spurious**" intersections. For example, consider the cube $ABCDA'B'C'D'$, where $ABCD$ is the bottom face and $A'B'C'D'$ is the top face. Although the lines BB' and CD *as drawn on the plane of projection* Π intersect, these lines are actually skew.

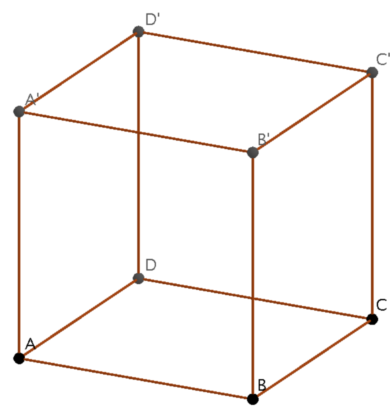


FIGURE 1.3. Cube $ABCDA'B'C'D'$.

However, because of Fact 1.4 above, we have the following crucial observation. If the projections of two lines *do* intersect, all we need to know to disambiguate the question of whether the actual lines interesect or not is to decide whether or not the lines lie in the same plane. For example, since we know that in the cube $ABCDA'B'C'D'$ the lines BB' and CD lie in two non-intersecting (parallel) planes (BB' lies in the front face $A'B'BA$ and CD lies in the back face $C'D'DC$), we know that the two lines BB' and CD do not actually intersect, and are thus skew.

We summarize this observation as the following:

Theorem 3. *If two distinct lines lie in the same plane and their projections intersect in a single point, then the lines intersect in a single point. If two lines do not lie in any single plane, they do not intersect even if their projections do.*

Problem 4. Prove Theorem 3.

Another very useful set of properties (many of which are usually taken to be axioms for three-dimensional geometry) are:

Fact 5. *The following hold:*

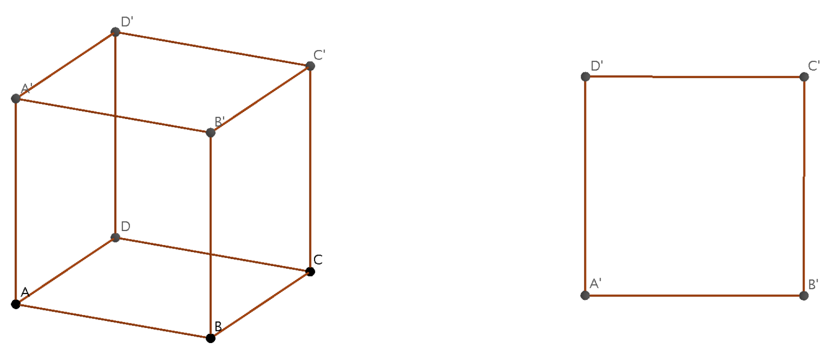
(1) *The intersection of any two distinct planes is either empty (if the planes are parallel) or is a line.*
(2) *Given any three distinct points, there is a unique plane through these points.*
(3) *Given any two distinct intersecting lines, there is a unique plane containing both of them.*
(4) *The intersection of any plane and a line is either empty (if the line is parallel to the plane), is the whole line (if the line lies in the plane), or consists of exactly one point.*
(5) *Given a line ℓ lying in a plane π and a point P in π but not on ℓ, there exists a unique line ℓ' which is parallel to ℓ and passes through P. Moreover, ℓ' also lies in the plane π .*

Problem 6. (a) Let $ABCDA'B'C'D'$ be a cube. Show that the diagonals $A'C$ and AC' intersect. Draw their point of intersection on a projection of the cube. (b) Show that D' belongs to the plane π passing through the points $AA'D$.

2. Some remarks on drawing figures

You may have noticed that the projection of a three-dimensional figure depends on the plane onto which we project and on the direction of the projection. Indeed, the shadow cast by an object depends both on the relative orientation of the object and the plane of the ground, as well as the direction of the sun's rays.

It is also rather obvious that some choices of the plane and line of projection give better and more insightful pictures than others. For example, the following two drawings are both projections of a cube (see Figure 2.1).



(A) (B)

Figure 2.1. Projections of a cube.

However, figure (a) is clearly much better than figure (b). Indeed, many distinct edges of the cube end up being projected to the same line in figure (b); some edges are even projected to single points!

It is important to draw pictures in which different lines project to different lines. This is always possible if one chooses the plane of projection correctly. We postpone the formulation and proof of this fact until the last section, since it is not essential for the remainder of our discussion.

Here are a few more tips about drawing figures:

- Draw large figures. It is OK to use the entire page for a figure.
- Draw neatly. Draw lines straight (use a straightedge if necessary). Avoid marks and corrections that may distract you: three dimensional figures are complicated enough already!
- Three-dimensional figures involve a lot of intersecting lines. Some of these lines actually intersect, and some are skew. Marking the true intersections (e.g. drawing a bold dot) helps you to imagine the picture correctly.
- It may help to draw lines that are behind other planes dotted.
- It may help to use a different color when drawing supplementary lines.
- Although using a computer is sometimes helpful, the point of the exercises in this handout is to get *you* to imagine and argue about three-dimensional pictures.

2.1. **On projections and distances.** It is not possible to find a projection from three-dimensional space to two-dimensional space with the property that the distance between any two points is the same as the distance between their projections. (Why?)

It is, however, possible to ask that distances be never increased.

Problem 7. Let P, Q be two points and let $\bar{P}$ and $\bar{Q}$ be their projections onto a plane Π along lines perpendicular to Π. Assume that the line of projection ℓ is perpendicular to Π. Show that the distance between $\bar{P}$ and $\bar{Q}$ is no bigger than the distance between P and Q.

Projections for which the line of projection ℓ is perpendicular to the plane of projection Π are called *orthogonal projections.* Projections which are not perpendicular are called *oblique.*

Problem 8. Show that an oblique projection along a direction that is *not* perpendicular to the plane Π there are some points P, Q for which the distance between their oblique projections is bigger than the distance between P and Q.

Taken together, the previous two problems characterize orthogonal projections as ones that are *contractive*: they never increase distances.

3. Sections

If S is a 3-dimensional figure (e.g., a tetrahedron or a cube) and π is a plane, then the intersection of π and S is called the the *section of S by the plane π.* The section will lie in the plane π and will in general be a polygon. Figure 3.1 shows a section of a tetrahedron.

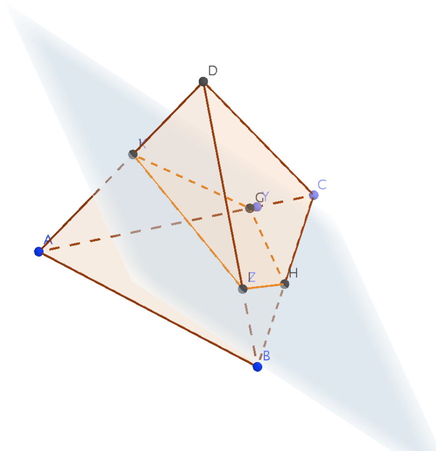


FIGURE 3.1. Section of a tetrahedron.

Problem 9. Let S be a (i) tetrahedron, (ii) cube. Describe all possible sections of S by planes.

3.1. **Drawing the section.** The first order of business is to find the intersection of the plane π with all of the faces of S. The idea is to proceed step by step, finding more intersections as we go.

Algorithm 10. *Let F_1 and F_2 be two faces of S intersecting along a common edge QR. Let us denote by ℓ_1 the intersection of π and F_1 and by ℓ_2 the intersection of π and F_2 (see Figure 3.2).*

Our algorithm will, allow us to, given ℓ_1 and one point W on ℓ_2, determine ℓ_2 as follows:

(1) *Extend ℓ_1 until it intersects QR. Call the point of intersection Y.*
(2) *Draw a line through W and Y. The intersection of this line with the face F_2 is ℓ_2.*

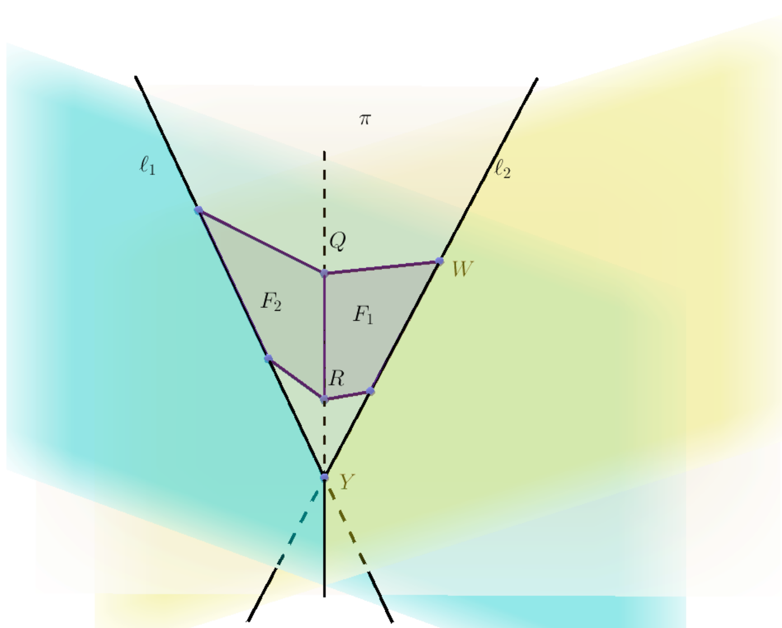
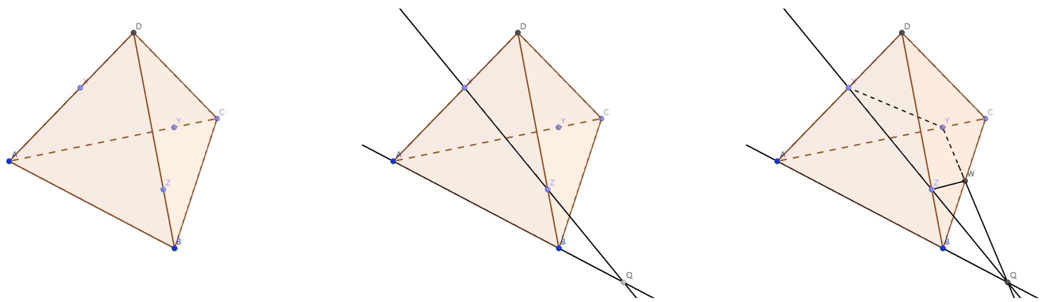


FIGURE 3.2. Finding the section using algorithm 10.

Problem 11. Explain why this works, i.e., why is UV the intersection of π with F_2?

Now let's use the algorithm to find the intersection of a tetrahedron with a plane. Let us suppose that the tetrahedron is as drawn



(A) (B) (C)

Figure 3.3. Find the section of a tetrahedron.

in Figure 3.3(a) and that the plane π is the plane through the points XYZ.

It is clear that all we are looking for is one extra point: the intersection of π with the edge BC.

The faces ABD and ABC share a common edge, AB. Moreover, we know the line segment which is the intersection of π and ABD (this is the segment XZ), and also a point on the intersection of ABC and π (the point Y). Thus we can apply our algorithm. We extend XZ until its intersection with the common edge between the two sections 3.3(b) and call this intersection Q. Then YQ is the line along which π intersects the plane ABC. So to find the remaining intersection point, we find the intersection of YQ with BC; we call this point W.

We have now found the section: it is the quadrilateral $XYZW$.

4. More examples

Problem 12. Consider the cube shown in Figure 4.1. Let π be the plane XYZ. Find the section of the cube by this plane.

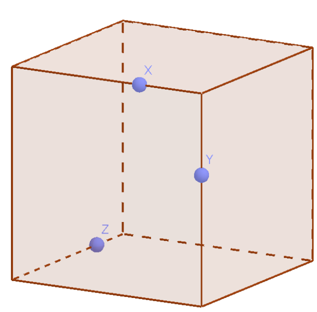


Figure 4.1. Figure for Problem 12.

Problem 13. Consider the cube shown in Figure 4.2. Let π be the plane XYZ. Find the section of the cube by this plane.

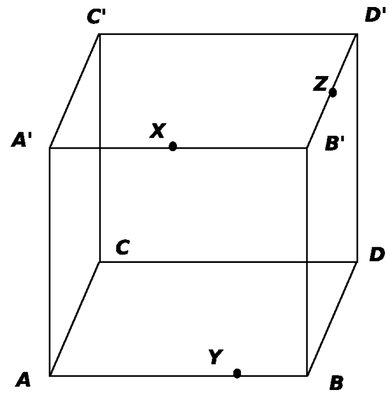


FIGURE 4.2. Figure for Problem 13.

However, sometimes we cannot use Algorithm 10, because we cannot find a single face for which we know the line at which π intersects it. Here is an example:

Problem 14. Consider the cube shown in Figure 4.3. Let π be the plane XYZ. Find the section of the cube by this plane.

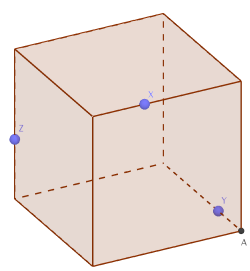


FIGURE 4.3. Figure for Problem 14.

5. More problems

These problems are taken from Ch. 15 of Textbook on mathematics, G.N. Yakovlev, ed., Nauka, Moscow, 1985 (in Russian).

Problem 15. Let $ABCD$ be a tetrahedron all of whose edges have length a. The edges DA, DC and BC contain points M, N and P, respectively, so that $|DM| = |CN| = a/3$, $|CP| = a/5$ (here $|\,|$ denotes length). Find the section of the tetrahedron by the plane MNP. Denote by Q the point at which this plane intersects the edge AB. Find the length of BQ.

Problem 16. Consider the pyramid $SABCD$ with base $ABCD$. Assume that $ABCD$ is a parallelogram (i.e., opposite edges are parallel). Let M and P be the midpoints of SB and SD. Find the intersection of the plane AMP with SC. Find the ratio of the line segments into which the plane divides SC.

Problem 17. Let $ABCDA'B'C'D'$ be a parallelepiped (i.e., opposite faces are parallel). Let M, N, P be points on AA', CC' and $C'D'$, respectively, so that $|AM| : |AA'| = |C'N| : |C'C| = |C'P| : |C'D'|$. Find the point Q at which the plane MNP intersects the line BC and find the ratio $|BQ| : |BC|$.

Problem 18. Let $ABCD$ be a tetrahedron. A plane is drawn through the vertex C and the midpoints of edges AD and BD. Find the ratio of the parts into which this plane divides the line segment MN, where M and N are the midpoints of the edges AB and CD.

6. Choosing the right plane of projection

Problem 19. Suppose that one is given n pairwise non-equal lines $\ell_1, \ldots, \ell_n$ in three-dimensional space. Show that there is always a choice of a two-dimensional plane so that the orthogonal projections $\bar{\ell_1}, \ldots, \bar{\ell_n}$ of these lines are pairwise non-equal lines.

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