Problem 1. Show that the equation
\[ 2^{2021} + (p - 2)^{2021} = 3^n \]
has no solutions where \( n \) is a positive integer and \( p \) is a prime.

Problem 2. Find the last digit of \( 3^{3\cdot3\cdot\cdots\cdot3} \), where there are 2021 three’s.

Problem 3. Let \( a, b, c, d \) be integers so that \( a^2 + b^2 + c^2 + d^2 = 2021 \). Show that at least one of them is congruent to 2 mod 4.

Problem 4.
(a) Show that a difference of two squares \( a^2 - b^2 \) cannot be congruent to 2 mod 4.
(b) Conversely, show that any integer \( n \not\equiv 2 (\text{mod} \ 4) \) can be written as the difference of two squares \( n = a^2 - b^2 \).

Problem 5. Show that
\[ 101 \mid 1^{10} + 2^{10} + \cdots + 100^{10}. \]
Hint: multiply the sum by \( 2^{10} \). What do you notice?

Problem *6. (Quadratic residues) Let \( p \) be an odd prime number.
(a) Show that there is at least one residue class \( \hat{a} \) mod \( p \) that is not a perfect square (that is, there are no integers \( b \) such that \( b^2 \equiv a \ (\text{mod} \ p) \)).
(b) Show that there are exactly \( \frac{p-1}{2} \) such residue classes \( \hat{a} \). Note: perfect squares mod \( p \) are called the quadratic residues.

Problem *7. (Multiplicative orders)
(a) Let \( p \) be a prime number, \( a \not\equiv 0 \ (\text{mod} \ p) \), and let \( n \) be the smallest positive integer such that \( p \mid a^n - 1 \). If \( m \) is another positive integer, show that \( p \mid a^m - 1 \) if and only if \( n \mid m \).
(b) Show that in (a), we have \( n \mid p - 1 \). Note: \( n \) is called the (multiplicative) order of \( a \) mod \( p \).
(c) Let \( k \) be a positive integer. Find all prime divisors \( p \) of \( 2^{2k} - 1 \) of the form \( p \equiv 3 \ (\text{mod} \ 4) \). Challenge: Can you find two (significantly different) solutions?

Homework

Problem 1. Let \( a, b, c \) be integers such that \( a^2 + b^2 = c^2 \). Show that \( 4 \mid ab \).

Problem 2. Does 101 divide
\[ 1^{10} + 2^{10} + \cdots + 50^{10}? \]