ORMC: MODULAR ARITHMETIC – PART II

OLYMPIAD GROUP 1, WEEK 5

Problem 1. Show that the equation
$$2^{2021} + (p-2)^{2021} = 3^n$$
has no solutions where n is a positive integer and p is a prime.

Problem 2. Find the last digit of $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$, where there are 2021 three's.

Problem 3. Let a, b, c, d be integers so that $a^2 + b^2 + c^2 + d^2 = 2021$. Show that at least one of them is congruent to 2 mod 4.

Problem 4.

(a) Show that a difference of two squares $a^2 - b^2$ cannot be congruent to 2 mod 4.

(b) Conversely, show that any integer $n \not\equiv 2 \pmod 4$ can be written as the difference of two squares $n = a^2 - b^2$.

Problem 5. Show that
$$101 \mid 1^{10} + 2^{10} + \cdots + 100^{10}.$$
Hint: multiply the sum by 2^{10}*. What do you notice?*

Problem *6. (Quadratic residues) Let p be an *odd* prime number.

(a) Show that there is at least one residue class $\hat{a}$ mod p that is not a perfect square (that is, there are no integers b such that $b^2 \equiv a \pmod p$).

(b) Show that there are exactly $\frac{p-1}{2}$ such residue classes $\hat{a}$. *Note: perfect squares mod p are called the quadratic residues.*

Problem *7. (Multiplicative orders)

(a) Let p be a prime number, $a \not\equiv 0 \pmod p$, and let n be the smallest positive integer such that $p \mid a^n - 1$. If m is another positive integer, show that $p \mid a^m - 1$ if and only if $n \mid m$.

(b) Show that in (a), we have $n \mid p-1$. *Note: n is called the (multiplicative) order of a mod p.*

(c) Let k be a positive integer. Find all prime divisors p of $2^{2^k} - 1$ of the form $p \equiv 3 \pmod 4$. *Challenge: Can you find two (significantly different) solutions?*

Homework

Problem 1. Let a, b, c be integers such that $a^2 + b^2 = c^2$. Show that $4 \mid ab$.

Problem 2. Does 101 divide
$$1^{10} + 2^{10} + \cdots + 50^{10}?$$