GRAPH COLORINGS

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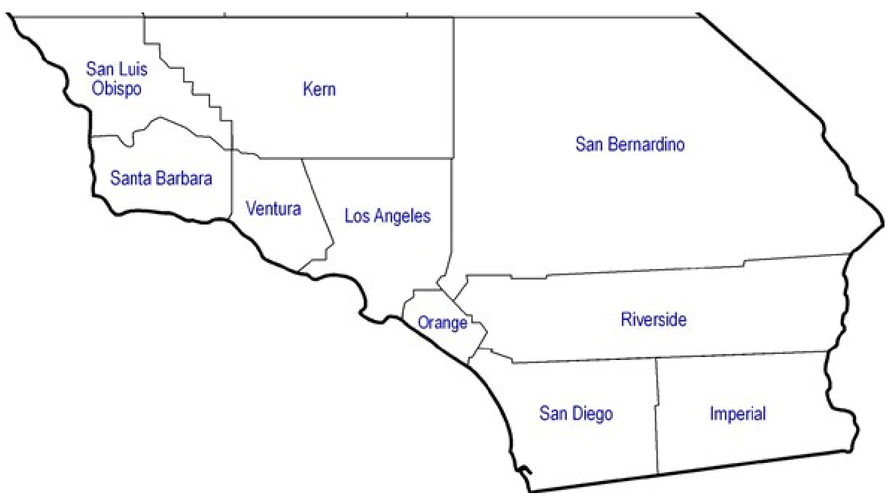
1 Introduction

Today's topics revolve around coloring graphs and maps.

Problem 1.

1. Using as few colors as possible, color the counties of southern California so that no two counties sharing a land border have the same color (corners that touch are not necessarily borders). Argue why using fewer colors is not possible.



2. Recall that a graph is a set of vertices connected by edges. How can we model a map by a graph in way that allows us to convert map coloring questions to a question about the graph?

Solution. You can color the map in 3 colors. There are groups of three counties for which each county borders each other county in the group. Such groups require at least 3 colors.

We can interpret a maps as a graph in two ways. One way is to interpret the corners of counties as vertices and the borders between them as edges. For us, a more useful interpretation will capture our method of coloring. We regard counties as vertices and draw an edge between vertices if the counties share a land border. Then coloring the map means coloring the vertices so that no two adjacent vertices have the same color. □

Note that not only can maps be represented by graphs, they are represented by *planar graphs*! This means we'll have all of the tools from the theory of planar graphs at our disposal when we study maps. In the second half of today, we'll explore the minimum number of colors needed to color maps, using the correspondence between maps and planar graphs.

To build up the theory of colorings, we will first discuss some generic examples coloring vertices of (possibly non-planar) graphs. A k-coloring of a graph G is a labelling of the graph's vertices with k colors such that no two vertices sharing the same edge have the same color. The smallest k such that a k-coloring exists is called the *chromatic number* of the graph, and denoted $\chi(G)$. The number of k-colorings of G is denoted $\chi_G(k)$, and it turns out that $\chi_G(k)$ is always a polynomial! We will prove this soon, but for now, let us explore this idea a little.

Problem 2. The path graph P_n has n vertices numbered $1, \ldots, n$ and $n-1$ edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$. Compute the chromatic number and polynomial of P_n.

Solution. When $n = 1$, the chromatic number $\chi(P_n) = 1$. When $n \geq 2$, the chromatic number $\chi(P_n) = 2$.

Start at a degree 1 vertex. Given k colors, we can color the first vertex any of the k colors. We may color the next vertex any of the remaining $k-1$ colors. This process repeats with each new vertex. Thus $\chi_{P_n}(k) = k(k-1)^{n-1}$. □

2 Chromatic polynomials

Before proving the main theorem of this section, which is that $\chi_G(k)$ is a polynomial, let's explore one more example.

Problem 3. The complete graph K_n has n vertices and an edge between any two distinct vertices. Compute the chromatic number and polynomial of K_n.

Solution. Since the complete graph has an edge between any two vertices, each vertex must be colored differently. Thus $\chi(K_n) \geq n$. Coloring each vertex a unique color is a valid coloring so $\chi(K_n) = n$.

Start at any vertex. Given k colors, we have k choices for color at the first vertex. No other vertex can have that color so we are left with $k-1$ choices for the next vertex. This process continues to obtain $\chi_{K_n}(k) = \prod_{i=0}^{n-1}(k-i)$. □

Now for the theorem.

Theorem 1. Let G be a graph and let $\chi_G : \mathbb{N} \to \mathbb{N}$ be the function where $\chi_G(k)$ is the number of distinct k-colorings of G. Then $\chi_G(k)$ is a polynomial.

Problem 4. *This problem requires a submission. As a group, discuss and formally write up a solution to this problem on a Google Doc. See instructor for details.*

We will prove the theorem as follows.

1. Let G be a graph and e be an edge of G. Denote by $G - e$ the graph obtained from G by deleting e (and leaving its two endpoints untouched). Denote by G/e the graph obtained from G by contracting e, i.e. deleting e, then merging its two endpoints into one. Explain why

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k).$$

2. We will use induction on the number of edges to prove the theorem. For the base case, show that $\chi_G(k)$ for the graph on n vertices with no edges is a polynomial by finding it explicitly.
3. Now assume that for all graphs with $m-1$ edges, $\chi_G(k)$ is a polynomial. Use part 1 to conclude that on graphs with m edges, that $\chi_G(k)$ is still a polynomial.

Solution.

1. We wish to compute $\chi_{G-e}(k) - \chi_G(k)$, which represents the number of k-colorings of $G - e$ that do not work as k-colorings for G. In these cases, the edge e connects two vertices of the same color. Thus this provides a valid k-coloring for G/e since the two vertices in question are merged. Similarly, given a valid k-coloring of G/e, we can extend this to a k-coloring of $G - e$ that does not work as a k-coloring of G by splitting the merged vertices. We conclude that $\chi_{G-e}(k) - \chi_G(k) = \chi_{G/e}(k)$.
2. We can pick any of the k colors for each vertex. Thus $\chi_G(k) = k^n$, a polynomial.
3. Let G be a graph on n vertices with m edges. Note that $G - e$ is a graph on n vertices with $m - 1$ edges and G/e is a graph on $n - 1$ vertices with $m - 1$ edges. The inductive hypothesis implies that $\chi_{G-e}(k)$ and $\chi_{G/e}(k)$ are polynomials. Then $\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$ is a polynomial by part 1.

□

To illustrate how the deletion-contraction idea behind the theorem can help us actually compute the chromatic polynomial of a more difficult graph, let us redo the path graph P_n as a toy example.

Problem 5. Find the chromatic polynomial of P_n again as follows:

1. In terms of the chromatic polynomial of P_{n-1}, what is the chromatic polynomial of P_n with the edge $\{1, 2\}$ deleted?
2. In terms of the chromatic polynomial of P_{n-1}, what is the chromatic polynomial of P_n with the edge $\{1, 2\}$ contracted?
3. Write out the recursion to find the chromatic polynomial of P_n.

Solution. 1. Let $e = \{1, 2\}$. Then $P_n - e$ will be P_{n-1} with an isolated vertex. We can color the isolated vertex with any of the k colors so $\chi_{P_n - e}(k) = k\chi_{P_{n-1}}(k)$.

2. We have P_n/e is P_{n-1} so $\chi_{P_n/e}(k) = \chi_{P_{n-1}}(k)$.

3. We note $P_1(k) = k$. By parts 1 and 2, $\chi_{P_n - e}(k) = k\chi_{P_{n-1}}(k)$ and $\chi_{P_n/e}(k) = \chi_{P_{n-1}}(k)$. By Problem 4, $\chi_{P_n}(k) = \chi_{P_n - e}(k) - \chi_{P_n/e}(k)$. Thus $\chi_{P_n} = (k-1)\chi_{P_{n-1}}$.

□

Problem 6. Find the chromatic number and polynomial of the following graphs (and explain your reasoning). Again, the deletion-contraction technique used in the proof of the theorem may help you, but you can use other methods if you wish.

1. Any tree T on n vertices. (Surprisingly, all trees on n vertices have the same chromatic polynomial! Recall that a tree is a graph that has no cycles, and that a tree with n vertices always has $n - 1$ edges.)

2. The cycle graph C_n on n vertices numbered $1, \ldots, n$ and exactly n edges being $\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}$.

Solution. 1. We will show via induction that $\chi_T(k) = k(k-1)^{n-1}$ where T is a tree on n vertices. Note that $\chi_T(k) = k$ when T has one vertex and $\chi_T(k) = \chi_{P_2}(k) = k(k-1)$ when T has two vertices. Let the tree T have n vertices. Then T will have at least one vertex with order one. Let e be the edge emanating from a vertex of order one. Then $T - e$ is a tree with $n - 1$ vertices and an isolated point while T/e is the same tree with $n - 1$ vertices. Denote the tree of $n - 1$ vertices as T'. Then $\chi_T(k) = (k-1)\chi_{T'}(k)$. By the inductive hypothesis, every tree on $n - 1$ vertices has $\chi_{T'}(k) = k(k-1)^{n-2}$. We conclude that $\chi_T(k) = k(k-1)^{n-1}$ for all trees T on n vertices.

2. Note that $\chi_{C_1}(k) = k$ and $\chi_{C_2}(k) = \chi_{P_2}(k) = k(k-1)$. Let $e = \{n, 1\}$. Then for $n \geq 3$, $C_n - e$ is P_n and C_n/e is C_{n-1}. We have $\chi_{C_n}(k) = \chi_{P_n}(k) - \chi_{C_{n-1}}(k)$. Thus $\chi_{C_n}(k) = \sum_{i=1}^{n}(-1)^{n+i}\chi_{P_i}(k)$. We have $\chi_{P_i}(k) = k(k-1)^{i-1}$ so

$$
\begin{aligned}
\chi_{C_n}(k) &= \sum_{i=2}^{n}(-1)^{n+i}k(k-1)^{i-1} \\
&= (-1)^n k(k-1)\left(\sum_{i=2}^{n}(-1)^i(k-1)^{i-2}\right) \\
&= (-1)^n k(k-1)\left(\frac{1 + (-1)^n(k-1)^{n-1}}{1 + (k-1)}\right) \\
&= (k-1)^n + (-1)^n(k-1).
\end{aligned}
$$

□

3 The five color theorem

We now return to the problem of coloring maps and planar graphs.

Problem 7.

1. Draw a map (or equivalently, a planar graph) that requires at least 4 colors.
2. Describe a graph that requires at least 100 colors. Is your graph planar? Can you draw a planar such graph?

Solution. K_4 is a planar graph that requires 4 colors, from a previous problem. K_{100} requires 100 colors but is not planar because it contains K_5 as a subgraph. □

As alluded to by the title of this section, we will prove the following theorem.

Theorem 2 (Five color theorem)**.** If G is a planar graph, then $\chi(G) \leq 5$.

Recall that the degree of a vertex is the number of edges coming out of it. Here is one fact that proof requires.

Problem 8. We will show that every planar graph has a vertex of degree at most 5.

1. Suppose for contradiction that every vertex has degree at least 6. Can you give a bound on the number of edges in the graph?
2. Recall that for planar graphs, $E \leq 3V - 6$. Use this to finish the proof.

Solution.

1. Let G have V vertices. Then the number of edges in G is at least $\frac{6V}{2} = 3V$.
2. We have $E \geq 3V$ by part 1. However, the inequality requires $E \leq 3V - 6$. This contradicts the assumption that each vertex has degree at least 6.

□

We can now prove the five color theorem!

Problem 9. We will use induction on the number of vertices.

1. The base case is when the graph has 1 vertex. Why is the five color theorem true in this case?
2. Now assume that all planar graphs on $n - 1$ vertices can be colored with 5 colors, and we want to show that all planar graphs on n vertices can too. Let v be a vertex of degree at most 5.
   (a) If v actually has degree at most 4, use the inductive hypothesis to quickly prove the claim.
   (b) Now suppose v has degree exactly 5, and like before, remove v and color the

graph by the inductive hypothesis. Why can we assume that the neighbors of v use all 5 colors?

(c) Suppose the 5 neighbors of v are colored (in clockwise order) red, yellow, green, blue, and purple. If there is no red-green alternating path between the red vertex and the green vertex, prove the claim. (Hint: try to swap colors around.)

(d) Suppose there is a red-green alternating path between the red vertex and the green vertex. Show that there is no yellow-blue alternating path between the yellow vertex and the blue vertex, and then finish the proof of the theorem.

Solution. 1. When a graph has one vertex, coloring the vertex any one color is a valid 5-coloring.

2. If v has degree at most 4, then the graph G with v and all adjacent edges removed has a 5-coloring by the inductive hypothesis. Apply the. 5-coloring to this part of G. The vertex v in G is adjacent to at most 4 other vertices. Thus there is a fifth color left over. We conclude that G has a 5-coloring.

   If the neighbors of v do not use all 5 colors, then the 5-coloring of G without v extends immediately to a 5-coloring of G.

   Assume that there is no alternating red-green path between the red and green vertex. Then we can switch the coloring of the green vertex to red. We switch the red neighbors of the former green vertex to green and so on. This process will provide a valid coloring that does not change the color of the original red vertex since there is no alternating path between the red and green vertices.

   Assume there is an alternating red-green path between the red and green vertices. In our planar drawing of G, the yellow vertex sits inside a cycle C where we go red vertex to green vertex via the assumed alternating path then green to v to red. The blue vertex sits outside the cycle in our planar drawing of G. Any alternating yellow-blue path from the yellow vertex to the blue vertex cannot intersect C. Thus we may apply the result of (c) to the yellow and blue vertices to obtain the result.

□

Today, we proved the five color theorem, but in fact, it turns out that four colors always suffice for planar graphs! In 1976, Kenneth Appel and Wolfgang Haken proved the *four color theorem*, which states that any planar graph can be colored with just 4 colors! Even though mathematicians have made progress in simplifying their original proof, which involved 1834 different cases to be checked by a computer, today's best known proof still involves 1492 cases to check.

4 Challenge: Ramsey theory

So far, we have been interested in coloring vertices, but what about edges? There are many ways that edge colorings can be defined, and in this section we will look at red-blue colorings, i.e. every edge is colored either red or blue. (There are also notions of edge coloring that map more closely to our vertex coloring, such as requiring that edges that meet the same vertex have different colors, but that's for another time.)

The number $R(r, s)$ called the Ramsey number for red r and blue s, and is the smallest n such that every red-blue edge coloring of K_n contains a red K_r or blue K_s as a subgraph. Recall that K_n is the complete graph on n vertices, meaning every pair of vertices is connected by an edge. One popular story to make this definition clearer is that $R(r, s)$ is the smallest n such that out of n people, we are guaranteed at least r to be mutual friends or s to be mutual strangers.

Problem 10. Find $R(2, n)$, for all $n \geq 1$. That is, what is the smallest group of people in which we are guaranteed at least 2 mutual friends or n mutual strangers?

Solution. Note that K_2 is made up of two vertices connected by an edge. Having any red edge in K_m forces a red K_2 to appear. Thus we must color each edge of K_n blue. Then a blue K_n will appear in K_m if and only if $m \geq n$. We conclude that $R(2, n) = n$. □

Problem 11. Show that $R(3, 3) = 6$. In order to do this, note that you need to show two things:

1. Explain why to prove $R(3, 3) > 5$, it suffices to construct an example red-blue coloring of K_5 that has neither no red triangles and no blue triangles. Then do it.
2. Explain why to prove $R(3, 3) \leq 6$, it suffices to find a red triangle or blue triangle in every red-blue coloring of K_6. Then do it.

Solution.

1. We will first show that $R(3, 3) > 5$. Picture K_5 as the pentagon with all edges in the interior. Then we can color K_5 so the all interior edges are blue and all edges in the pentagon are red. Any triangle requires at least one interior edge and one outer edge. As a result, it contains no K_3 of either color.
2. We will show that $R(3, 3) \leq 6$. Color the edges of K_6. By the pigeonhole principle, the vertex v must have at least three edges of the same color. Without loss of generality, assume these edges are red. Label the three corresponding vertices a, b, and c. Then if at least one of $\{a, b\}$, $\{b, c\}$, or $\{a, c\}$ is red, we have a red triangle. If all three edges are blue, we have a blue triangle. Thus K_6 will always contain either a red or blue K_3.

□

One fact that is not immediately clear is whether or not $R(r, s)$ even exists for all natural numbers r and s. In other words, is it necessarily true that you are guaranteed monochromatic complete subgraphs of any size, as long as you take the original graph to be much larger? In 1928, Frank Ramsey proved that you are indeed guaranteed this, and we will show it now too.

Theorem 3 (Ramsey's theorem)**.** $R(r, s)$ exists for all natural numbers r, s.

Problem 12. We will prove the theorem in the following steps:

1. Explain why it suffices to show the following three facts: (1) $R(r, 1)$ and $R(1, s)$ exist for all natural numbers r and s, and (2) $R(r, s) \leq R(r-1, s) + R(r, s-1)$.
2. Show (1) by computing them explicitly.
3. Show (2) by doing the following: Consider the complete graph on $R(r-1, s) + R(r, s-1)$ vertices. Pick any vertex v. Then let the set A contain all vertices connected to v by a red edge, and let B contain all the vertices connected to v by a blue edge.
   (a) Explain why either $|A| \geq R(r-1, s)$ or $|B| \geq R(r, s-1)$. (For any set S, $|S|$ denotes the number of elements in S.)
   (b) Assume that $|A| \geq R(r-1, s)$. Find a complete red subgraph with r vertices or a complete blue subgraph with s vertices.
   (c) Repeat the argument if $|B| \geq R(r, s-1)$ and conclude the theorem.

Solution. 1. This upper bounds $R(r, s)$ for all r, s by induction.

2. Both are always equal to 1.
3. (a) By construction of A and B, we have $R(r-1, s) + R(r, s-1) = |A| + |B| + 1$. The result follows.
   (b) Because $|A| \geq R(r-1, s)$, it either contains a complete blue subgraph on s vertices, in which case we are good, or a complete red subgraph on $r-1$ vertices. In this second case, we can add v to make a complete red subgraph on r vertices.
   (c) This direction is similar to the previous one. □

Computing $R(r, s)$ is difficult for even moderately large r and s. As of today, mathematicians do not even know the value of $R(5, 5)$.

Problem 13. Find the best upper and lower bounds on $R(5, 5)$ that you can.

Solution. We can show that $R(5, 5) \geq 10$. Take K_9 and group the vertices into two groups of 5 vertices $\{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$ and $\{v_{21}, v_{22}, v_{23}, v_{24}\}$. Then we can color all edges red except the complete bipartite graph between groups and a full cycle within the group of 5. Take any 5 vertices. Then there is a blue edge among these 5 vertices by construction. A K_5 requires 10 edges. The only way we get 10 or more blue edges is by taking three or more vertices from the group of 5. In the three case, however, the two vertices from the group of

4 are connected by a red edge. In the four case, there are red vertices among the four chosen from the group of 5. In the five case, there are many red vertices among the five.

We can also show $R(5,5) \leq 70$. Recall that we know $R(2,n) = R(n,2) = n$. Now, we apply the inequality from the previous problem to fill in the following table:

	2	3	4	5
2	2	3	4	5
3	3	6	10	15
4	4	10	20	35
5	5	15	35	70

Note that this actually generates Pascal's triangle as an upper bound! □

The late mathematician Paul Erdős, who was one of the greatest mathematicians of the 20th century, once said the following: "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

More than just calculating values of $R(r,s)$, Ramsey theory is an active field of mathematics research where we try to find order amidst chaos. For instance, no matter how chaotic the red-blue coloring of K_6 may be, we can still find order because it must contain a monochromatic K_3. In general, this field is about asking: how large should objects be in order to guarantee the existence of a subobject satisfying some property?