UCLA MATH CIRCLE FALL 2011
PIGEONHOLE PRINCIPLE BY ALI GUREL

Most of the problems are selected from the books Problem Solving Strategies by Arthur Engel and Principles and Techniques in Combinatorics by Chuan-Chong Chen and Khee-Meng Koh.

(1) Show that there is a number in the set
$$\{2^1 - 1, 2^2 - 1, \ldots, 2^{98} - 1\}$$
which is divisible by 99.

(2) Among 51 integers from $\{1, 2, \ldots, 100\}$ there are two which are coprime.

(3) Let $A \subset \{1, 2, \ldots, 100\}$ such that $|A| = 51$. Show that there exist $a, b \in A$, with $a \neq b$ such that $a \mid b$.

(4) Twenty pairwise distinct positive integers are all < 70. Prove that among their pairwise differences there are four equal numbers.

(5) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

(6) Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $1/8$.

(7) Let A be a set of 6 points in a plane such that no 3 are collinear. Show that there exist 3 points in A which form a triangle having an interior angle not exceeding $30°$.

(8) A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but no more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

(9) (Spanish MO-1989) Fifteen problems, numbered 1 through 15, are posed on a certain examination. No student answers two consecutive problems correctly. If 1600 candidates sit the test, must at least two of them have the identical answer patterns? (Assume each question has only 2 possible answers, right or wrong, and assume that no student leaves any question unanswered.

(10) For a finite set A of integers, denote by $s(A)$ the sum of numbers in A. Let S be a subset of $\{1, 2, 3, \ldots, 14, 15\}$ such that $s(B) \neq s(C)$ for any 2 disjoint subsets B, C of S. Show that $|S| \leq 5$.

1. Challenge Problems.

(1) (USAMO-1985) There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them, each of whom either knows both or else knows neither of the two. Assume that knowing is a symmetric relation, and that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(2) (IMO-1978) An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.