## GRAPH COLORINGS

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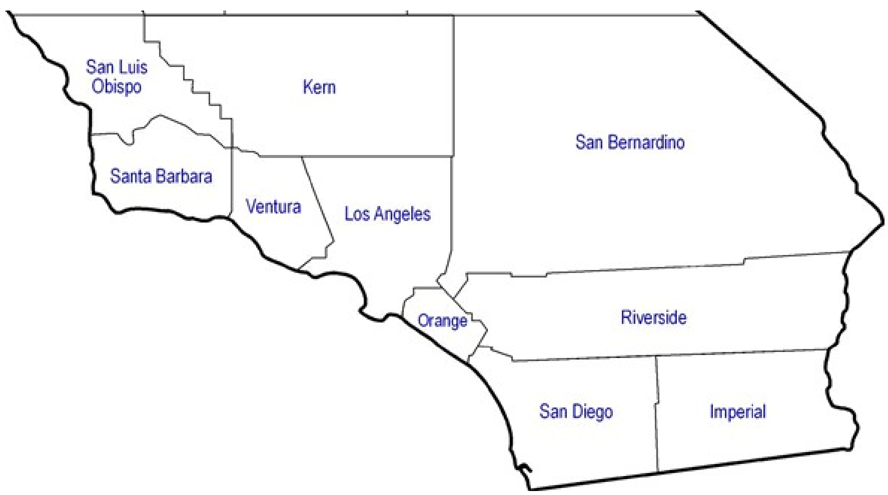
# 1 Introduction

Today's topics revolve around coloring graphs and maps.

**Problem 1.**

1. Using as few colors as possible, color the counties of southern California so that no two counties sharing a land border have the same color (corners that touch are not necessarily borders). Argue why using fewer colors is not possible.




2. Recall that a graph is a set of vertices connected by edges. How can we model a map by a graph in way that allows us to convert map coloring questions to a question about the graph?

Note that not only can maps be represented by graphs, they are represented by *planar graphs*! This means we'll have all of the tools from the theory of planar graphs at our disposal when we study maps. In the second half of today, we'll explore the minimum number of colors needed to color maps, using the correspondence between maps and planar graphs.

To build up the theory of colorings, we will first discuss some generic examples coloring vertices of (possibly non-planar) graphs. A $k$-coloring of a graph $G$ is a labelling of the graph's vertices with $k$ colors such that no two vertices sharing the same edge have the same color. The smallest $k$ such that a $k$-coloring exists is called the *chromatic number* of the graph, and denoted $\chi(G)$. The number of $k$-colorings of $G$ is denoted $\chi_G(k)$, and it turns out that $\chi_G(k)$ is always a polynomial! We will prove this soon, but for now, let us explore this idea a little.

**Problem 2.** The path graph $P_n$ has $n$ vertices numbered $1, \ldots, n$ and $n-1$ edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$. Compute the chromatic number and polynomial of $P_n$.

# 2 Chromatic polynomials

Before proving the main theorem of this section, which is that $\chi_G(k)$ is a polynomial, let's explore one more example.

**Problem 3.** The complete graph $K_n$ has $n$ vertices and an edge between any two distinct vertices. Compute the chromatic number and polynomial of $K_n$.

Now for the theorem.

**Theorem 1.** Let $G$ be a graph and let $\chi_G : \mathbb{N} \to \mathbb{N}$ be the function where $\chi_G(k)$ is the number of distinct $k$-colorings of $G$. Then $\chi_G(k)$ is a polynomial.

**Problem 4.** *This problem requires a submission. As a group, discuss and formally write up a solution to this problem on a Google Doc. See instructor for details.*

We will prove the theorem as follows.

1. Let $G$ be a graph and $e$ be an edge of $G$. Denote by $G - e$ the graph obtained from $G$ by deleting $e$ (and leaving its two endpoints untouched). Denote by $G/e$ the graph obtained from $G$ by contracting $e$, i.e. deleting $e$, then merging its two endpoints into one. Explain why

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k).$$

2. We will use induction on the number of edges to prove the theorem. For the base case, show that $\chi_G(k)$ for the graph on $n$ vertices with no edges is a polynomial by finding it explicitly.

3. Now assume that for all graphs with $m-1$ edges, $\chi_G(k)$ is a polynomial. Use part 1 to conclude that on graphs with $m$ edges, that $\chi_G(k)$ is still a polynomial.

To illustrate how the deletion-contraction idea behind the theorem can help us actually compute the chromatic polynomial of a more difficult graph, let us redo the path graph $P_n$ as a toy example.

**Problem 5.** Find the chromatic polynomial of $P_n$ again as follows:

1. In terms of the chromatic polynomial of $P_{n-1}$, what is the chromatic polynomial of $P_n$ with the edge $\{1, 2\}$ deleted?
2. In terms of the chromatic polynomial of $P_{n-1}$, what is the chromatic polynomial of $P_n$ with the edge $\{1, 2\}$ contracted?
3. Write out the recursion to find the chromatic polynomial of $P_n$.

**Problem 6.** Find the chromatic number and polynomial of the following graphs (and explain your reasoning). Again, the deletion-contraction technique used in the proof of the theorem may help you, but you can use other methods if you wish.

1. Any tree $T$ on $n$ vertices. (Surprisingly, all trees on $n$ vertices have the same chromatic polynomial! Recall that a tree is a graph that has no cycles, and that a tree with $n$ vertices always has $n - 1$ edges.)
2. The cycle graph $C_n$ on $n$ vertices numbered $1, \ldots, n$ and exactly $n$ edges being $\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}$.

# 3 The five color theorem

We now return to the problem of coloring maps and planar graphs.

**Problem 7.**

1. Draw a map (or equivalently, a planar graph) that requires at least 4 colors.
2. Describe a graph that requires at least 100 colors. Is your graph planar? Can you draw a planar such graph?

As alluded to by the title of this section, we will prove the following theorem.

**Theorem 2** (Five color theorem)**.** If $G$ is a planar graph, then $\chi(G) \leq 5$.

Recall that the degree of a vertex is the number of edges coming out of it. Here is one fact that proof requires.

**Problem 8.** We will show that every planar graph has a vertex of degree at most 5.

1. Suppose for contradiction that every vertex has degree at least 6. Can you give a bound on the number of edges in the graph?
2. Recall that for planar graphs, $E \leq 3V - 6$. Use this to finish the proof.

We can now prove the five color theorem!

**Problem 9.** We will use induction on the number of vertices.

1. The base case is when the graph has 1 vertex. Why is the five color theorem true in this case?
2. Now assume that all planar graphs on $n-1$ vertices can be colored with 5 colors, and we want to show that all planar graphs on $n$ vertices can too. Let $v$ be a vertex of degree at most 5.
   (a) If $v$ actually has degree at most 4, use the inductive hypothesis to quickly prove the claim.
   (b) Now suppose $v$ has degree exactly 5, and like before, remove $v$ and color the graph by the inductive hypothesis. Why can we assume that the neighbors of $v$ use all 5 colors?
   (c) Suppose the 5 neighbors of $v$ are colored (in clockwise order) red, yellow, green, blue, and purple. If there is no red-green alternating path between the red vertex and the green vertex, prove the claim. (Hint: try to swap colors around.)
   (d) Suppose there is a red-green alternating path between the red vertex and the green vertex. Show that there is no yellow-blue alternating path between the yellow vertex and the blue vertex, and then finish the proof of the theorem.

Today, we proved the five color theorem, but in fact, it turns out that four colors always suffice for planar graphs! In 1976, Kenneth Appel and Wolfgang Haken proved the *four color theorem*, which states that any planar graph can be colored with just 4 colors! Even though mathematicians have made progress in simplifying their original proof, which involved 1834 different cases to be checked by a computer, today's best known proof still involves 1492 cases to check.

# 4 Challenge: Ramsey theory

So far, we have been interested in coloring vertices, but what about edges? There are many ways that edge colorings can be defined, and in this section we will look at red-blue colorings, i.e. every edge is colored either red or blue. (There are also notions of edge coloring that map more closely to our vertex coloring, such as requiring that edges that meet the same vertex have different colors, but that's for another time.)

The number $R(r, s)$ called the Ramsey number for red $r$ and blue $s$, and is the smallest $n$ such that every red-blue edge coloring of $K_n$ contains a red $K_r$ or blue $K_s$ as a subgraph. Recall that $K_n$ is the complete graph on $n$ vertices, meaning every pair of vertices is connected by an edge. One popular story to make this definition clearer is that $R(r, s)$ is the smallest $n$ such that out of $n$ people, we are guaranteed at least $r$ to be mutual friends or $s$ to be mutual strangers.

**Problem 10.** Find $R(2, n)$, for all $n \geq 1$. That is, what is the smallest group of people in which we are guaranteed at least 2 mutual friends or $n$ mutual strangers?

**Problem 11.** Show that $R(3, 3) = 6$. In order to do this, note that you need to show two things:

1. Explain why to prove $R(3, 3) > 5$, it suffices to construct an example red-blue coloring of $K_5$ that has neither no red triangles and no blue triangles. Then do it.
2. Explain why to prove $R(3, 3) \leq 6$, it suffices to find a red triangle or blue triangle in every red-blue coloring of $K_6$. Then do it.

One fact that is not immediately clear is whether or not $R(r, s)$ even exists for all natural numbers $r$ and $s$. In other words, is it necessarily true that you are guaranteed monochromatic complete subgraphs of any size, as long as you take the original graph to be much larger? In 1928, Frank Ramsey proved that you are indeed guaranteed this, and we will show it now too.

**Theorem 3** (Ramsey's theorem)**.** $R(r, s)$ exists for all natural numbers $r, s$.

**Problem 12.** We will prove the theorem in the following steps:

1. Explain why it suffices to show the following three facts: (1) $R(r, 1)$ and $R(1, s)$ exist for all natural numbers $r$ and $s$, and (2) $R(r, s) \leq R(r - 1, s) + R(r, s - 1)$.
2. Show (1) by computing them explicitly.
3. Show (2) by doing the following: Consider the complete graph on $R(r - 1, s) + R(r, s - 1)$ vertices. Pick any vertex $v$. Then let the set $A$ contain all vertices connected to $v$ by a red edge, and let $B$ contain all the vertices connected to $v$ by a blue edge.
   (a) Explain why either $|A| \geq R(r - 1, s)$ or $|B| \geq R(r, s - 1)$. (For any set $S$, $|S|$ denotes the number of elements in S.)
   (b) Assume that $|A| \geq R(r - 1, s)$. Find a complete red subgraph with $r$ vertices or a complete blue subgraph with $s$ vertices.
   (c) Repeat the argument if $|B| \geq R(r, s - 1)$ and conclude the theorem.

Computing $R(r, s)$ is difficult for even moderately large $r$ and $s$. As of today, mathematicians do not even know the value of $R(5, 5)$.

**Problem 13.** Find the best upper and lower bounds on $R(5, 5)$ that you can.

The late mathematician Paul Erdős, who was one of the greatest mathematicians of the 20th century, once said the following: "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

More than just calculating values of $R(r, s)$, Ramsey theory is an active field of mathematics research where we try to find order amidst chaos. For instance, no matter how chaotic the red-blue coloring of $K_6$ may be, we can still find order because it must contain a monochromatic $K_3$. In general, this field is about asking: how large should objects be in order to guarantee the existence of a subobject satisfying some property?