Trigonometry and Complex Numbers - Euler's Formula

Richard Yim

24 January 2021

1 Warm-up (Before We Put it All Together)

Here are some warm-up problems related to the topics that we'll be exploring today.

1.1 Complex Numbers

1. Let $z_1 = a_1 - b_1 i$ and $z_2 = a_2 + b_2 i$. What is the real part of the complex number $z_1 + z_2$ [$Re(z_1 + z_2)$]? What is the imaginary part of the complex number $z_1 + z_2$ [$Im(z_1 + z_2)$]?

2. With the same z_1 and z_2 from above, what is $z_1 \cdot z_2$ as a complex number in standard form (e.g. $Re(z_1 \cdot z_2) + Im(z_1 \cdot z_2)i$)?

3. What is $\frac{a+bi}{c+di}$ in standard form with no complex number in the denominator? (Hint: "rationalizing" complex numbers; remember the complex conjugate).

4. Define the "modulus of a complex number" $z = a + bi$ to be

$$|z| = \sqrt{z \cdot \bar{z}}$$

and recall that if z is a complex number we define the "conjugate of a complex number" to be the switch of sign of the imaginary part of the complex number:

$$\bar{z} = \overline{a + bi} = a - bi$$

Show that

$$|z| = \sqrt{a^2 + b^2}$$

(Food for thought: What is interesting about this formula and the Pythegorean theorem? We will find out together later in the handout...)

1.2 Trigonometry

1. Suppose $\cos\theta = \sin\alpha$. What values of α is this true? Since this expression is very general there is more than one possible value for α (Hint: in terms of π and θ).

2. Suppose $\cos\theta = \cos\beta$. What are some values for which β is this true? Since this expression is very general there is more than one possible value for β (Hint: symmetric functions; periodic functions).

3. What values of β_1 and β_2 (with $\beta_1 \neq \beta_2$) is $\sin(\alpha + \beta_1) = \sin(\alpha + \beta_2)$?

4. What values of γ_1 and γ_2 (with $\gamma_1 \neq \gamma_2$) is $\sin(\alpha + \gamma_1) = \sin(\alpha + \gamma_2)$?

1.3 Exponentiation

1. Let $10 \cdot 10 \cdot 10 = 10^n$. What is n?

2. Let $25 \cdot 25 \cdot 43 \cdot 48 \cdot 48 \cdot 13 = 25^{n_1} \cdot 48^{n_2} \cdot 43^{n_3} \cdot 13^{n_4}$. What are n_1, n_2, n_3 and n_4 equal to?

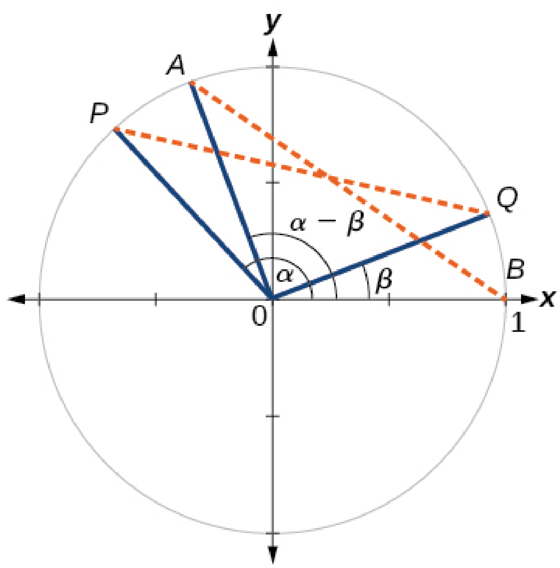
3. Let $b \cdot b \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} = b^{m_1} \cdot a^{m_2}$. What are m_1 and m_2 equal to? (Hint: exponents are not all necessarily positive.)

4. Let e be a constant, just a number (in fact, let be Euler's number: 2.718281828...) Let $e^a \cdot e^b = e^{p_1}$. What is p_1 equal to? What about $\frac{e^a}{e^b} = e^{p_2}$; what is p_2?

2 Angle Summation (Difference)

We pick up right where we left off from last week, which were the angle difference formulas. We start with cosine angle differences.

$$\cos(\alpha - \beta) = ?$$



1. The diagram above is the digram that we will be working with. The idea is to use the fact that we can relate the Pythagorean theorem to two different triangles, but have a result where the form give us equivalent answers. To be more precise, in our diagram triangle PQO is a counterclockwise rotation up until the positive x-axis, and has dimensions equal to triangle ABO.

2. First, we find the distance between points P and Q.

(a) The difference in height between points P and Q is what in terms of the sine of α and the sine of β?

(b) The difference in width between points P and Q is what in terms of the cosine of α and the cosine of β?

(c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

3. Now we find the distance between points A and B.

   (a) The difference in height between points A and O is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points A and O is what in terms of the cosine of α and the cosine of β?

(c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

4. Finally, you have formulas for the lengths of $\overline{PQ}$ and $\overline{AB}$. They are both equal to each other. Solve for $\cos(\alpha - \beta)$.

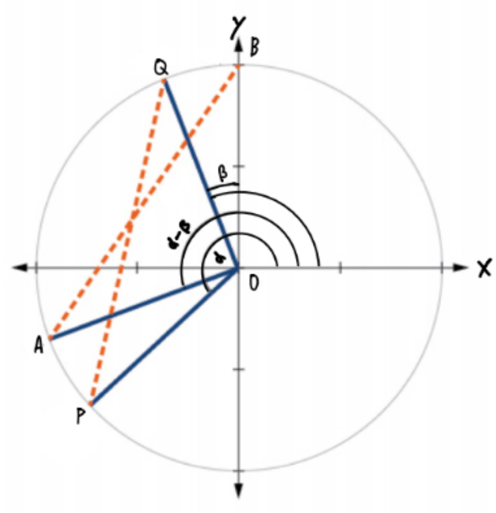
The above expression that you derived works generally for sums as well, where $\beta < 0$. In such a case, you should be aware of what happens to the sine of β (antisymmetric: $\sin(-\theta) = -\sin(\theta)$.

First, knowing that $\sin(-\theta) = -\sin(\theta)$, it must be that $\sin(\theta) = -\sin(-\theta)$. Then, with the formula for $\cos(\alpha - \beta)$, what is $\cos(\alpha + \beta)$ (write it below)?

Now we take a look at the sine angle difference (summation) formula; the process is very similar for how we got the cosine angle difference (summation) formula.

$$\sin(\alpha - \beta) = ?$$

Let's look at another figure that should help us figure this out (next page).



1. First, we find the distance between points P and Q.

   (a) The difference in height between points P and Q is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points P and Q is what in terms of the cosine of α and the cosine of β?

   (c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

2. Now we find the distance between points A and B (Hint: you may have to use the cosine difference formula you had derived earlier).

   (a) The difference in height between points A and O is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points A and O is what in terms of the cosine of α and the cosine of β?

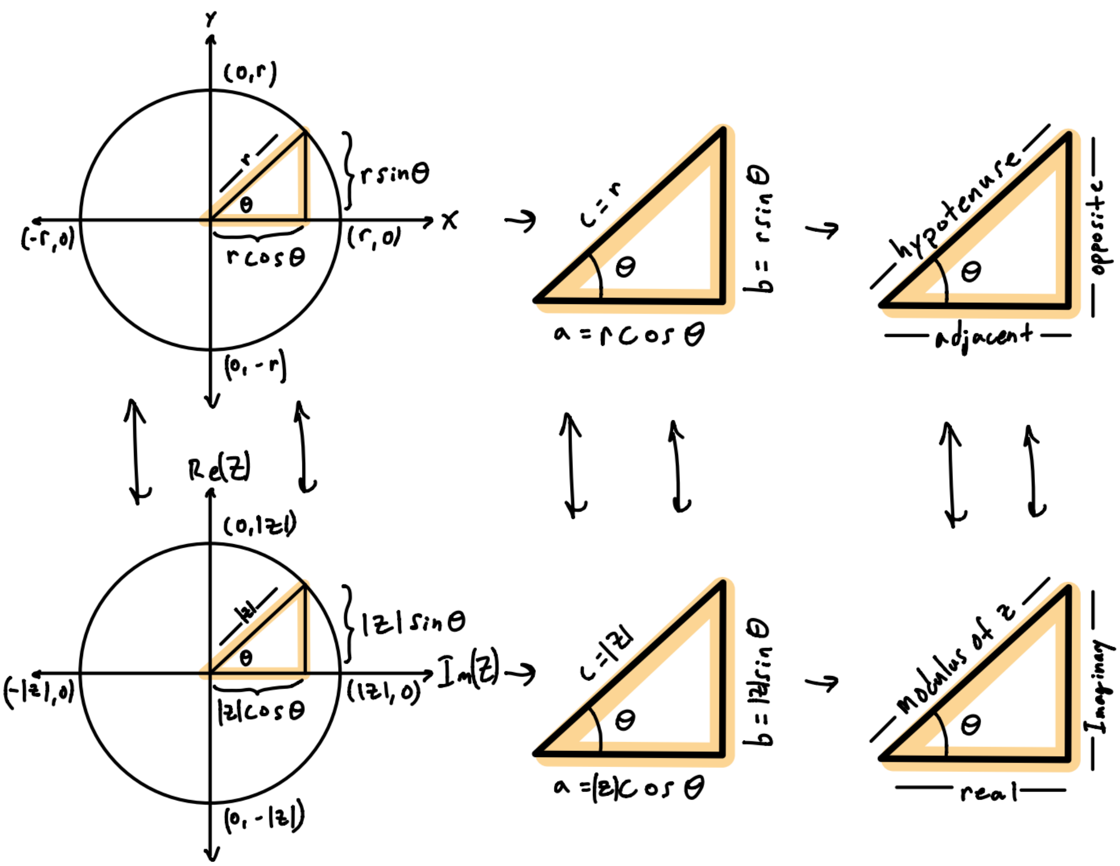
   (c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

3. Finally, you have formulas for the lengths of $\overline{PQ}$ and $\overline{AB}$. They are both equal to each other. Solve for $\sin(\alpha - \beta)$.

So now we've found $\sin(\alpha - \beta)$. What is this formula for $\sin(\alpha + \beta)$? (Hint: sine is an antisymmetric function. How did you find $\cos(\alpha + \beta)$ with $\cos(\alpha - \beta)$; can you use the same process here?) Write the formula below.

3 Euler's Formula

We asked in the warm-up, what is interesting about $|z|$ if $z = a + bi$. Well, it's in essentially the Pythagoren theorem for complex numbers in the complex plane! Take a look:



Explicitly,

$$|z|^2 = |z|^2 \cos^2 \theta + |z|^2 \sin^2 \theta \iff r^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta.$$

We call the figure in the bottom left of the figure, the "complex plane," and a number in that complex plane is defined a point on that plane. For example, if your complex number is $z = 1 + 2i$, then on the complex plane the number[complex] $1 + 2i$ lies 1 away from the origin (to the right of) along the "x-axis (real part of complexes)" and 2 above the origin along the "y-axis (imaginary part of complexes)."

Finally, there is a nice formula discovered by Leonhard Euler in the 1700s that allows us to relate complex numbers, trigonometric functions *and* exponents into one single formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Where e is known as "Euler's number" and has the following form

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots \approx 2.718281828...$$

The number e is an irrational number. The actual derivation of the formula requires some high school calculus (don't worry we'll leave that up to your future teachers).

Instead, we'll be interested in applying this formula to give some real answers to some questions we've had two weeks ago as well make some of our trigonometric derivations easier.

3.1 Square Root of the Square Root of -1

We answer the following: what is $\sqrt{\sqrt{-1}} = \left((-1)^{\frac{1}{2}}\right)^{\frac{1}{2}} = (-1)^{\frac{1}{4}} = \sqrt{i}$?

1. Using Euler's formula, for what value of θ is

$$e^{i\theta} = i?$$

2. Using your θ, θ' from (1) we know that

$$(e^{i\theta'})^{\frac{1}{2}} = i^{\frac{1}{2}} = \sqrt{i} \implies \sqrt{i} = e^{i\left(\frac{\theta'}{2}\right)}$$

   How can we rewrite $e^{i\left(\frac{\theta'}{2}\right)}$ using Euler's formula?

3. So what is $\sqrt{i}$ equal to as a complex number in standard form?

   (Challenge question: What is $i^{\frac{1}{n}}$, for any n? What if n is negative? What if it's not an integer? What if n gets infinitely positively large or infinitely negative?)

3.2 Writing Cosine and Sine in Terms of Euler's Formula

Here we find a way to write Euler's formula in terms of sine and cosine and the imaginary number i.

3.2.1 Cosine

1. Knowing that $e^{i\theta} = \cos\theta + i\sin\theta$, what is $e^{i\theta} + e^{-i\theta}$?

2. Using the answer from above, what do you have to divide for what value of a is

$$\frac{e^{i\theta} + e^{-i\theta}}{a} = \cos\theta?$$

3.2.2 Sine

1. Knowing that $e^{i\theta} = \cos\theta + i\sin\theta$, what is $e^{i\theta} - e^{-i\theta}$?

2. Using the answer from above, what do you have to divide for what value of b is

$$\frac{e^{i\theta} + e^{-i\theta}}{a} = \sin\theta?$$

   (Hint: is a necessarily a real number?)

3.3 Angle Summation with Euler's Formula

We derive the angle summation formula's using Euler's Formula.

3.3.1 Cosine Summation

1. We found in the previous subsection that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

   What is $\cos(\alpha + \beta)$ using this formula? You already know the formula from your trigonometry derivation, so your answer should match; however, the point of this exercise is to see if it's possible to derive it using Euler's formula. By definition it is, but you should still show these things. Hint: $e^{i(a+b)} = e^{ia}e^{ib}$.

3.3.2 Sine Summation

1. We found in the previous subsection that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

   What is $\sin(\alpha + \beta)$ using this formula? You already know the formula from your trigonometry derivation, so your answer should match; however, the point of this exercise is to see if you can derive it using Euler's formula. By definition it is, but you should still show these things. Hint: $e^{i(a+b)} = e^{ia}e^{ib}$.

(You can use the back of this page for more workspace.)