GRAPH COLORINGS

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OLGA RADKO MATH CIRCLE ADVANCED 2

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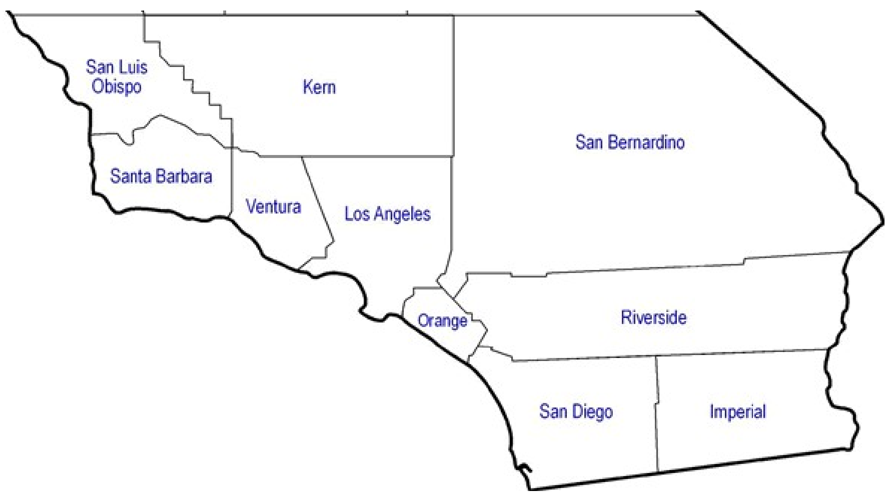
1 Introduction

Today's topics revolve around coloring graphs and maps.

Problem 1.

1. Using as few colors as possible, color the counties of southern California so that no two counties sharing a land border have the same color (corners that touch are not necessarily borders). Argue why using fewer colors is not possible.



2. Recall that a graph is a set of vertices connected by edges. How can we model a map by a graph in way that allows us to convert map coloring questions to a question about the graph?

Note that not only can maps be represented by graphs, they are represented by *planar graphs*! This means we'll have all of the tools from the theory of planar graphs at our disposal when we study maps. In the second half of today, we'll explore the minimum number of colors needed to color maps, using the correspondence between maps and planar graphs.

To build up the theory of colorings, we will first discuss some generic examples coloring vertices of (possibly non-planar) graphs. A k-coloring of a graph G is a labelling of the graph's vertices with k colors such that no two vertices sharing the same edge have the same color. The smallest k such that a k-coloring exists is called the *chromatic number* of the graph, and denoted $\chi(G)$. The number of k-colorings of G is denoted $\chi_G(k)$, and it turns out that $\chi_G(k)$ is always a polynomial! We will prove this soon, but for now, let us explore this idea a little.

Problem 2. The path graph P_n has n vertices numbered $1, \ldots, n$ and $n-1$ edges being $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$. Compute the chromatic number and polynomial of P_n.

2 Chromatic polynomials

Before proving the main theorem of this section, which is that $\chi_G(k)$ is a polynomial, let's explore one more example.

Problem 3. The complete graph K_n has n vertices and an edge between any two distinct vertices. Compute the chromatic number and polynomial of K_n.

Now for the theorem.

Theorem 1. Let G be a graph and let $\chi_G : \mathbb{N} \to \mathbb{N}$ be the function where $\chi_G(k)$ is the number of distinct k-colorings of G. Then $\chi_G(k)$ is a polynomial.

Problem 4. *This problem requires a submission. As a group, discuss and formally write up a solution to this problem on a Google Doc. See instructor for details.*

We will prove the theorem as follows.

1. Let G be a graph and e be an edge of G. Denote by $G - e$ the graph obtained from G by deleting e (and leaving its two endpoints untouched). Denote by G/e the graph obtained from G by contracting e, i.e. deleting e, then merging its two endpoints into one. Explain why

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k).$$

2. We will use induction on the number of edges to prove the theorem. For the base case, show that $\chi_G(k)$ for the graph on n vertices with no edges is a polynomial by finding it explicitly.

3. Now assume that for all graphs with $m-1$ edges, $\chi_G(k)$ is a polynomial. Use part 1 to conclude that on graphs with m edges, that $\chi_G(k)$ is still a polynomial.

To illustrate how the deletion-contraction idea behind the theorem can help us actually compute the chromatic polynomial of a more difficult graph, let us redo the path graph P_n as a toy example.

Problem 5. Find the chromatic polynomial of P_n again as follows:

1. In terms of the chromatic polynomial of P_{n-1}, what is the chromatic polynomial of P_n with the edge $\{1, 2\}$ deleted?
2. In terms of the chromatic polynomial of P_{n-1}, what is the chromatic polynomial of P_n with the edge $\{1, 2\}$ contracted?
3. Write out the recursion to find the chromatic polynomial of P_n.

Problem 6. Find the chromatic number and polynomial of the following graphs (and explain your reasoning). Again, the deletion-contraction technique used in the proof of the theorem may help you, but you can use other methods if you wish.

1. Any tree T on n vertices. (Surprisingly, all trees on n vertices have the same chromatic polynomial! Recall that a tree is a graph that has no cycles, and that a tree with n vertices always has $n - 1$ edges.)
2. The cycle graph C_n on n vertices numbered $1, \ldots, n$ and exactly n edges being $\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}$.

3 The five color theorem

We now return to the problem of coloring maps and planar graphs.

Problem 7.

1. Draw a map (or equivalently, a planar graph) that requires at least 4 colors.
2. Describe a graph that requires at least 100 colors. Is your graph planar? Can you draw a planar such graph?

As alluded to by the title of this section, we will prove the following theorem.

Theorem 2 (Five color theorem)**.** If G is a planar graph, then $\chi(G) \leq 5$.

Recall that the degree of a vertex is the number of edges coming out of it. Here is one fact that proof requires.

Problem 8. We will show that every planar graph has a vertex of degree at most 5.

1. Suppose for contradiction that every vertex has degree at least 6. Can you give a bound on the number of edges in the graph?
2. Recall that for planar graphs, $E \leq 3V - 6$. Use this to finish the proof.

We can now prove the five color theorem!

Problem 9. We will use induction on the number of vertices.

1. The base case is when the graph has 1 vertex. Why is the five color theorem true in this case?

2. Now assume that all planar graphs on $n-1$ vertices can be colored with 5 colors, and we want to show that all planar graphs on n vertices can too. Let v be a vertex of degree at most 5.

   (a) If v actually has degree at most 4, use the inductive hypothesis to quickly prove the claim.

   (b) Now suppose v has degree exactly 5, and like before, remove v and color the graph by the inductive hypothesis. Why can we assume that the neighbors of v use all 5 colors?

   (c) Suppose the 5 neighbors of v are colored (in clockwise order) red, yellow, green, blue, and purple. If there is no red-green alternating path between the red vertex and the green vertex, prove the claim. (Hint: try to swap colors around.)

   (d) Suppose there is a red-green alternating path between the red vertex and the green vertex. Show that there is no yellow-blue alternating path between the yellow vertex and the blue vertex, and then finish the proof of the theorem.

Today, we proved the five color theorem, but in fact, it turns out that four colors always suffice for planar graphs! In 1976, Kenneth Appel and Wolfgang Haken proved the *four color theorem*, which states that any planar graph can be colored with just 4 colors! Even though mathematicians have made progress in simplifying their original proof, which involved 1834 different cases to be checked by a computer, today's best known proof still involves 1492 cases to check.