ORMC: INEQUALITIES

OLYMPIAD GROUP 1, WEEK 2

[*Recap: Triangle similarity*]

Ceva's Theorem. Let $\triangle ABC$ be a triangle, and let D, E, F be points inside the segments BC, CA, AB respectively. Then the

$$\text{Lines } AD, BE, CF \text{ are concurrent} \qquad \textit{if and only if} \qquad \frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1.$$

Menelaus' Theorem. Let $\triangle ABC$ be a triangle. Let E, F lie inside the segments CA respectively AB, and let D lie on the line BC such that C lies between B and D. Then the

$$\text{Points } D, E, F \text{ are collinear} \qquad \textit{if and only if} \qquad \frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1.$$

Problem 1. Prove the ($\Rightarrow$) implication in Menelaus' Theorem by constructing a point X on the line FD such that $XC \parallel AB$, and using three triangle similarities.

Problem 2. Prove the ($\Rightarrow$) implication in Ceva's Theorem by applying Menelaus' theorem twice (that's all you need!).

Problem 3. Let $\triangle ABC$ be a triangle.

(a) Prove that the three medians $\triangle ABC$ are concurrent using Ceva's theorem.

(b) Let A_1, A_2 split the segment BC into thirds (so $BA_1 = A_1A_2 = A_2C = BC/3$). Define B_1, B_2 on CA and C_1, C_2 on AB analogously. Which of the lines $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$ are concurrent?

Problem 4. Let $\triangle ABC$ be a triangle.

(a) Let $D \in BC$ be the foot of the angle bisector of $\angle BAC$. Show that $BD/BA = CD/CA$.

Hint: Draw a parallel through D to AB.

(b) Show that the three angle bisectors of $\triangle ABC$ are concurrent.

Problem 5.

(a) Let $\triangle ABC$ be an acute triangle. Prove that the heights in $\triangle ABC$ are concurrent using Ceva's Theorem.

(b*) Let $\triangle ABC$ be an obtuse triangle such that $\angle BAC > 90°$. Prove that the heights in $\triangle ABC$ are concurrent using Menelaus' Theorem.

Hint: Let $\{H_1\} := AD \cap BE$ and $\{H_2\} := AD \cap CF$. Apply Menelaus' Theorem in $\triangle H_1BD$ and $\triangle H_2CD$ to conclude that $AH_1 = AH_2$.

Problem 6. Let $\triangle ABC$ be equilateral with side length 3. Let X lie inside the segment AB such that $AX = 1$, and let Y be such that B, C, Y are collinear in this order with $CY = 1$. Prove that XY is perpendicular to AB.

Homework

Problem 1. In a triangle $\triangle ABC$, let X, Y lie inside the segments AB respectively AC, such that XY is parallel to BC. Let Z be the intersection of BY and CX. Show that AZ is the median line from A of $\triangle ABC$.

Problem 2. Let $\triangle ABC$ be equilateral with side length 3. Let E, F lie inside the segments CA respectively AB, and let D lie on the line BC such that C lies between B and D. Suppose that D, E, F are collinear and $AF = CE = CD = x$. Find x.

Hint: Write out Menelaus' Theorem and express every length therein in terms of x. There's also a solution using a $30^\circ - 60^\circ - 90^\circ$ triangle.