Introduction to Trigonometry

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1 Trigonometry

Before continuing on to the next natural topic related to complex numbers, we need to build some language with trigonometry (trigonometry and complex numbers are very conveniently well connected).

Trigonometry is the part of math that is obsessed with triangle side lengths and angles. We will be only interested in right triangles, triangles with exactly one angle having 90 degress.

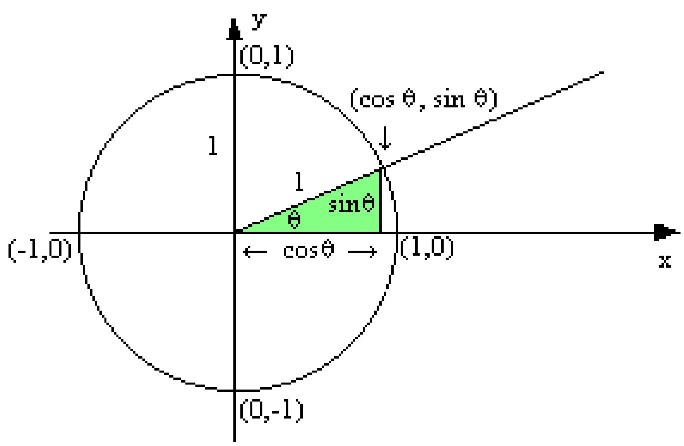


Figure 1: What is sine and cosine anyways?

The two main functions to discuss relationships between the angles and side lengths of right triangles are the *sine* and *cosine* functions. The sine and cosine functions are functions that express the relationship beteen the interior angle of a triangle and the side lengths of the triangle. Take a look at figure 1. We see that θ is the angle we are interested in, located at the origin. Then we write the "sine of theta" as $\sin\theta$; similarly, we write the "cosine of theta" as $\cos\theta$.

But what are the definitions of sine and cosine exactly? $\sin\theta$ is defined as the ratio of the side length opposite to theta over length of the hypotenuse; $\cos\theta$ is defined as the ratio of the side length adjacent to theta over the length of the hypotenuse. We distinguish the adjacent angle from the hypotenuse, as the hypotense is the longest side length of a right triangle by the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

Here is a more visual explanation for what $\sin\theta$ and $\cos\theta$ really are:

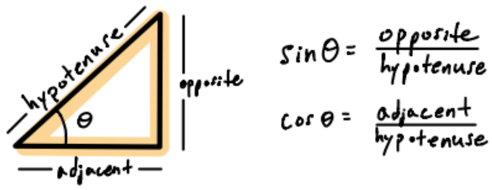


Figure 2: Definition of sine and cosine.

Let's look at this formula with respect to what we see on the unit circle in figure 1:

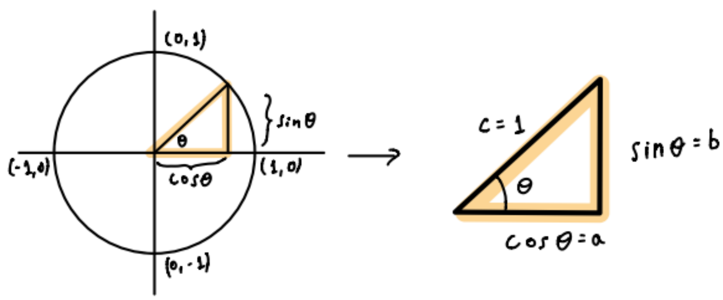


Figure 3: Pythagorean theorem, sine and cosine.

We notice that we have a right triangle inscribed in the circle. What else do we notice? the hypotenuse of such triangle, if always inscribed in the circle and centered at the origin, will have exactly one right angle at all times along with a hypotenuse that is always length 1 (as the radius of the unit circle is always 1; the radius *is* the hypotenuse in this case). Then, we see that we can use the Pythagorean theorem (relating the side lengths of a right traingle) to arrive at the following equation:

$$a^2 + b^2 = c^2 \implies \cos^2\theta + \sin^2\theta = 1$$

Also, as notation, we don't write $(\cos\theta)^2$, but $\cos^2\theta$. Another important bit of convention (what is normally done) is that the angle theta increases in a counteclockwise fashion (from quadrant to quadrant, up, left, down, right, and up again). More generally for any point on a plane, we can imagine that point lying on a circle of radius r. We can again image a right triangle reaching that point in the plane. In such a general case, what would the Pythagorean theorem along with our new functions $\sin\theta$ and $\cos\theta$ look like? (next page)

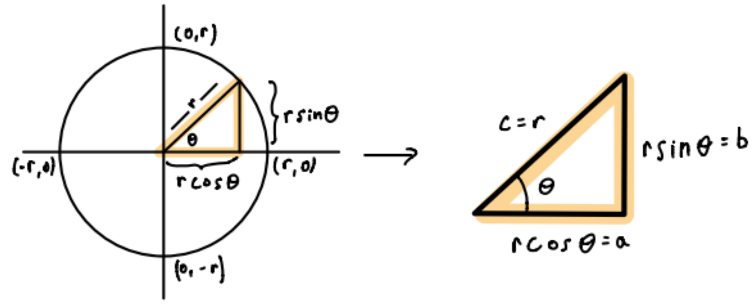


Figure 4: Pythagorean theorem, sine and cosine.

Looking at figure 4 we see that we can define the values a and b by using our definitions of sine and cosine along with the radius. Precisely, we can describe any point (a, b) in terms of the distance from the origin (radius) as well as the angle made from the first line where θ starts, and

$$(a, b) = (r\cos\theta, r\sin\theta)$$

All points on the plane can in fact be described with just the radius and the angle θ (theta). In fact, we can use the pythagorean theorem and to find how far a point (a, b) is from the origin:

$$a^2 + b^2 = c^2 \implies c = \sqrt{a^2 + b^2}$$

Below are some basic questions to see if you understand these definitions.

1. What is the sine and cosine of an angle? What is $\sin(45°)$ and $\cos(45°)$.

2. Suppose there is a ladder leaning up against a building just barely reaching the roof. The base of the ladder is 1.5 meters from the building, and the building is 10 meters tall. How long must the ladder be?

3. Suppose you are 100 meters from your home and 45 degrees away from the street where home is located. If your home is the "origin" what is your current location in (x, y)-coordinates?
   (a) Begin by drawing a diagram of what this problem is describing (your home, origin; your location, point; the angle from the street, theta).
   (b) Using your diagram write what position you are in (x,y)-coordinates (you do not have to find the exact number, and you can leave your answer in terms of sine and cosine).

1.1 Radians and Degrees

What is θ (theta) exactly? We know that θ is an angle. But theta can be understood in two different ways. The angle θ can be understood in degrees as well as radians. That's where the definition of π comes in. *pi* is the proportion for which diameter of a circle equals its circumference (the circumference is π times the diameter). π is what is known as an irrational number, as in you cannot express it as the ratio of two integers.

Furthermore, the definition of π is what's known as a radian. We can go between radians and degrees by the using the definition of π. In fact, π is equal to exactly 180 degrees.

$$\pi(\text{radians}) = 180°(\text{degrees})$$

We call "regular" real numbers "radians" and we can go between radians and degrees with the following formula:

$$180° = \pi \implies 1° = \left(\frac{180}{\pi}\right) \implies \alpha(\text{radians})\frac{180}{\pi} = \alpha(\text{degrees})$$

To go from degrees to radians we use the following:

$$\pi = 180° \implies \left(\frac{\pi}{180}\right) = 1 \implies \alpha(\text{degrees})\frac{\pi}{180} = \alpha(\text{radians})$$

One more important thing: we say that angles are "periodic." Suppose that you are at sitting on a circle and you go around it twice. How many degrees have you gone across the edge of the circle? Since there are 360 degrees in a circle, the first go around is 360 degrees; the second go around is another 360 degrees, for a total of 720 degrees. But, you're right back where you started. So really, your position, with respect to degrees, is just 0. Here are some simple questions to see if you understand these definitions:

1. What is 9π radians in degrees?

2. What is 140° in radians?

3. What is the "actual" number of degrees if your angle θ is 361°? What if you are at π radians and you travel 2π radians around a circle. What is your angle in radians? (Hint: periodic return)

1.2 Cosine and Sine Functions

Now we explore what sine and cosine are more generally. Sine and cosine, with their geometric motivation, are simply functions, or well-defined "maps" that takes "one thing" to "another thing." So we say that the sine function takes an angle to a real number, where the real number is in fact a ratio between the opposite side and the hypotenuse. This is similarly true for the cosine function, but for adjacent sides by hypotenuses.

If you haven't noticed, so far we've only been really dealing with sine and cosine when the angle θ is less than 90°. What if the angle is greater than 90°? What if it's almost 360°? In the following set, we look at what happens when the "theta" of sine and cosine is not between 0 and 90 degrees.

1. First draw a two perpendicular axes (a normal plane with an origin). Place a point in the first quadrant; label that point (x_1, y_1). What is the sign of x_1? The sign of y_1?

2. Earlier we found a way to relate the coordinate of a point with its distance from the origin as well as the angle from the positive x-axis. How can we rewrite (x_1, y_1)? Write it in terms of r_1 and θ_1 (no exact numbers just label them so). Can the "radius" or distance from the origin ever be negative?

3. Now place another point in quadrant II. Label it (x_2, y_2). Write this coordinate in terms of r_2 and θ_2. What is the sign of x_2? What is the sign of $r_2 \cos\theta_2$?

4. Now place another point in quadrant III. Label it (x_3, y_3). Write this coordinate in terms of r_3 and θ_3. What is the sign of x_3 and y_3? What is the sign of $r_3 \cos\theta_3$ and $r_3 \sin\theta_3$

5. Just one more time! Place another point in quadrant IV. Label it (x_4, y_4). Write this coordinate in terms of r_4 and θ_4. What is the sign of x_4 and y_4? What is the sign of $r_3 \sin\theta_4$?

6. What do you notice about the sine and cosine functions? Since r_1, r_2, r_3 and r_4 are just distances, can we ever have the these distances are negative? More specifically, these are really "hypotenuses," so can these hypotenuse lengths be negative? What does this mean for sine and cosine?

From the previous set you should have notice that and take on negative values! This means that as we change the angle, depending the angle itself, sine and cosine will take on positive and negative values. In fact, we already know that sine and cosine will be always at most 1, and at least -1. Why? We already showed the following:

$$\sin^2\theta + \cos^2\theta = 1$$

This is again true by the Pythagorean theorem, imagining that we have a triangle with hypotenuse length 1:

$$\sin^2\theta + \cos^2\theta = 1 \iff \left(\frac{opposite}{hypotenuse}\right)^2 + \left(\frac{adjacent}{hypotenuse}\right)^2 = 1$$

and,

$$\left(\frac{opposite}{hypotenuse}\right)^2 + \left(\frac{adjacent}{hypotenuse}\right)^2 = 1 \iff (opposite)^2 + (adjacent)^2 = (hypotenuse)^2$$

Finally, notice that this formula is really independent of whether the sides are positive or negative since we are squaring the "lengths." More generally the Pythagorean theorem is just a relationship with the magnitude of distances three points that form a right triangle. We make the last remark: the big thing to understand is that sine and cosine are both at most 1 and at least -1. Now, instead of putting general points on a plane, let's put these points on a unit circle in the next set...

1. First draw the unit circle.

2. Let's take a look at the cosine function. What is the sign of cosine in the first quadrant? Second, third, fourth?

3. As you go around the unit circle what values does cosine take on? Do the values repeat?

4. What is the sign of sine in the first quadrant? Second, third, fourth?

5. As you go around the unit circle what values does sine take on? Do the values repeat?

6. What values of β_1 and β_2 (with $\beta_1 \neq \beta_2$) is $\sin(\alpha + \beta_1) = \sin(\alpha + \beta_2)$?

   What values of γ_1 and γ_2 (with $\gamma_1 \neq \gamma_2$) is $\cos(\alpha + \beta_1) = \cos(\alpha + \beta_2)$?

   How many such β_1 and β_2, and γ_1 and γ_2 are there? What do you notice about the difference between γ_1 and γ_2 as well as β_1 and β_2?

This “repeating” of the sine and cosine functions is known as periodicity, and we say that these functions are ”periodic functions.” From here are on out we make some major distinctions. When we talk about the ”angle” of the triangle, or the angle of ”sine” and ”cosine,” we only think of the angle in terms of ”radians.” Again, we can relate radians to degrees as we’ve shown in the previous section. Once again, radians are the real number counterparts to degrees, and we also found in the previous section that radians, with respect to the sine and cosine functions are ”periodic” or repeat. In the figure above, we have the

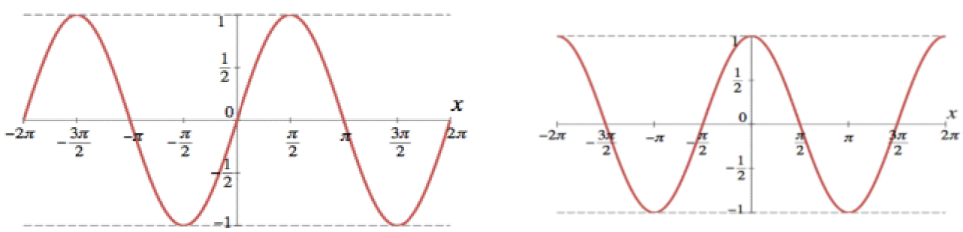


Figure 5: (left) plot of sine function. (right) plot of cosine function.

plot of the sine and cosine functions and their tendencies to repeat along the real number line (radians) from negative to positive infinity, while being bounded between -1 and 1. We also notice that the ”shift” between sine and cosine:

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

You can notice this between our unit circle in the previous section as we saw that the value of sine seemed to always be ”catching up” to the cosine, starting from 0 and that cosine had a $\frac{\pi}{2}$ head start. Finally, we also describe sine to be antisymmetric and cosine to be symmetric (you can check this out for yourself).

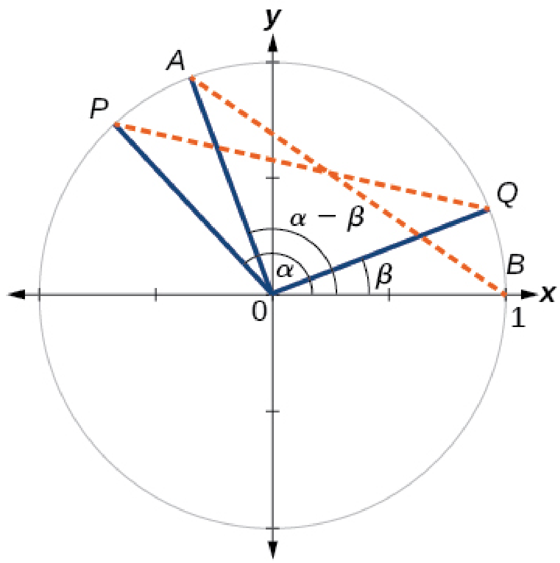
$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(-\theta) = \cos(\theta)$$

1.3 Angle Summation (Difference)

In this section we now take a look at two very important and useful formulas. We start with cosine angle differences.

$$\cos(\alpha - \beta) = ?$$

Let’s start on a new page.



1. The diagram above is the digram that we will be working with. The idea is to use the fact that we can relate the Pythagorean theorem to two different triangles, but have a result where the form give us equivalent answers. To be more precise, in our diagram traingle PQO is a counterclockwise rotation up until the positive x-axis, and has dimensions equal to triangle ABO.

2. First, we find the distance between points P and Q.

   (a) The difference in height between points P and Q is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points P and Q is what in terms of the cosine of α and the cosine of β?

(c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

3. Now we find the distance between points A and B.

   (a) The difference in height between points A and O is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points A and O is what in terms of the cosine of α and the cosine of β?

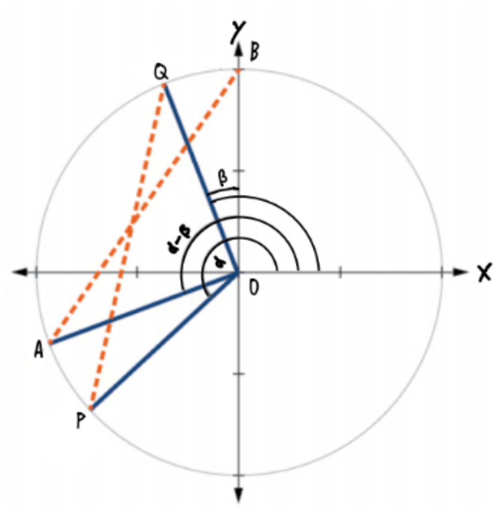
   (c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

4. Finally, you have formulas for the lengths of $\overline{PQ}$ and $\overline{AB}$. They are both equal to each other. Solve for $\cos(\alpha - \beta)$.

The above expression that you derived works generally for sums as well, where $\beta < 0$. In such a case, you should be aware of what happens to the sine of β (antisymmetric: $\sin(-\theta) = -\sin(\theta)$.

Now we take a look at the sine angle summation formula; the process is very similar for how we got the cosine angle summation formula.

$$\sin(\alpha - \beta) = ?$$



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(c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

2. Now we find the distance between points A and B (Hint: you may have to use the cosine difference formula you had derived earlier).

   (a) The difference in height between points A and O is what in terms of the sine of α and the sine of β?

   (b) The difference in width between points A and O is what in terms of the cosine of α and the cosine of β?

(c) Using the differences you expressed above, what is the length of line PQ? (Hint: Pythagorean theorem)

3. Finally, you have formulas for the lengths of $\overline{PQ}$ and $\overline{AB}$. They are both equal to each other. Solve for $\sin(\alpha - \beta)$.

Great job! You've derived the difference formulas for both cosine and sine. As mentioned earlier, you can easily derive the regular summation fromula by switching some signs around your formula and accounting for the fact that the sine function is antisymmetric. Go ahead and write the angle sum formulas below using the angle difference formulas you have derived.