

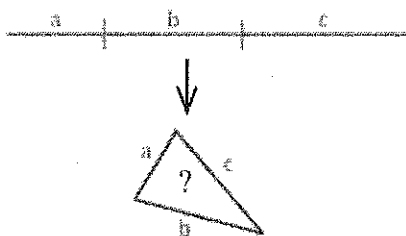
(SOLUTIONS)

BREAKING STICKS

INTERMEDIATE GROUP - MARCH 12, 2017

Warm Up

Suppose that Nikola has a straight stick of length 1 and wants to break it into three pieces at random. He hopes that he can form a triangle with the sticks.



Consider the following two scenarios:

- Nikola simultaneously breaks the stick, at random, into three pieces.
- Nikola breaks the stick, at random, into two pieces and then he breaks the right piece at random again.

(1) Do you think the two scenarios are the same? Why or why not?

Answers may vary

(2) Consider the probability that those pieces form a triangle of positive area. Do you think the probabilities for the two scenarios are equal? If not, which do you think is larger?

Answers may vary

We will be looking at some triangle geometry and probability theory to test your hypothesis about the two questions above.

Handout based on January 2017 Cool Math Essay by James Tanton.

http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_January-2017_Sticks-and-Triangles.pdf.

Handout compiled by Ivy Wang/Los Angeles Math Circle/UCLA Department of Mathematics.

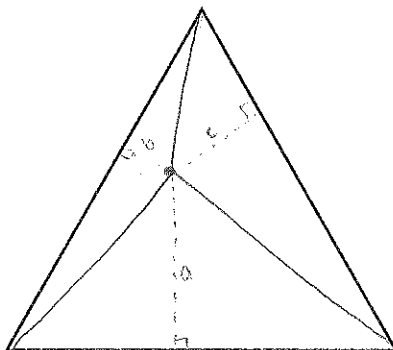
Viviani's Theorem

In order to solve the problem above, we first need to learn about Viviani's Theorem.

Theorem 1. *Vincenzo Viviani (1622 – 1703)* For any point inside an equilateral triangle, the sum of its distances from each of the three sides is constant and equal to the height of the triangle.

Problem 1. Proving Viviani's Theorem.

Suppose we have an equilateral triangle with side length s and height h .



- (1) Pick any point inside the equilateral triangle and connect the point to the three vertices of the triangle to form three smaller triangles.
- (2) Let the height of three smaller triangles be a , b , and c . Use dotted lines to indicate the altitudes of the smaller triangles from the point and label them accordingly.
- (3) We will show that $a + b + c = h$:

(a) What are the areas of each of the smaller triangles in terms of a, b, c , and s ?

$$\frac{1}{2}as, \quad \frac{1}{2}bs, \quad \frac{1}{2}cs$$

(b) What is the area of the equilateral triangle in terms of h and s ?

$$\frac{1}{2}hs$$

(c) Write a formula relating the sum of the areas of the three smaller triangles to the area of the larger triangle. Simplify your equation.

$$\frac{1}{2}as + \frac{1}{2}bs + \frac{1}{2}cs = \frac{1}{2}s(a+b+c) = \frac{1}{2}sh$$

$$(a+b+c) = h.$$

Applications of Viviani's Theorem

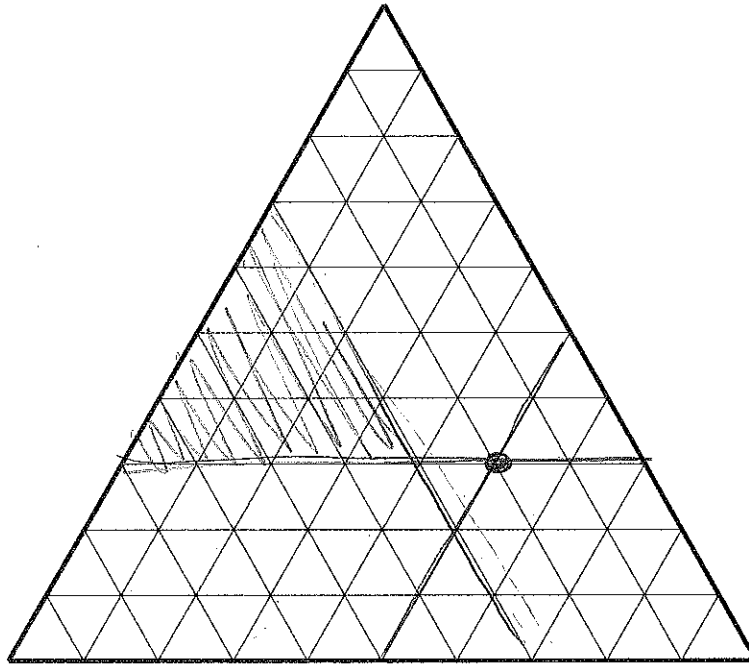
Problem 2. Benjamin lives in Triangle Land, which has the geography of an equilateral triangle. The three borders of Triangle Land are called the Southern border, Northeastern Border and Northwestern border. The shortest distance from the most northern point of triangle land to the Southern border is 10 miles.

- (1) Benjamin's house is 3 miles from the Southern border and 5 miles from the Northwestern border. Using Viviani's theorem, determine how far is he from the Northeastern border.

$$3 + 5 + d = 10$$

$$d = 2$$

- (2) Plot the location of Benjamin house on the grid below.



- (3) Triangle Land is building a wall along its Southern and Northeastern borders. The noise that this makes is too loud and Benjamin would like to move to an area that is at least 3 miles away from where the wall is being built. Shade in the area where Benjamin can potentially move to in the map above.

Breaking Sticks and Triangle Inequality

In this section, we will look at the conditions that need to be satisfied for the three resulting sticks to form a triangle.

Suppose Nikola broke the stick of length 1 into three pieces of length a , b , and c .



Problem 3. What are the constraints that a , b , and c must satisfy so that they can form a triangle?

- (1) What is the sum of a , b , and c ?

$$a + b + c = 1$$

- (2) Notice that the three sticks will not form a triangle when one of the sticks is too long and the other two sticks cannot “reach” each other. Thus, in order for the sticks to form a triangle, the following system of inequalities must be satisfied:

$$\begin{cases} a < b + c \\ b < a + c \\ c < a + b \end{cases}$$

These conditions are known as the triangle inequality

- (3) Add the necessary terms to each of the three inequalities above so that the left hand side of each is $a + b + c$.

$$\begin{cases} 2a < a + b + c \\ 2b < a + b + c \\ 2c < a + b + c \end{cases}$$

- (4) Replace $a + b + c$ in each of the three inequalities with the value you wrote in part (1).

$$\begin{cases} 2a < \frac{1}{2} \\ 2b < \frac{1}{2} \\ 2c < \frac{1}{2} \end{cases}$$

- (5) Simplify the system of inequalities.

$$\begin{cases} a < \frac{1}{4} \\ b < \frac{1}{4} \\ c < \frac{1}{4} \end{cases}$$

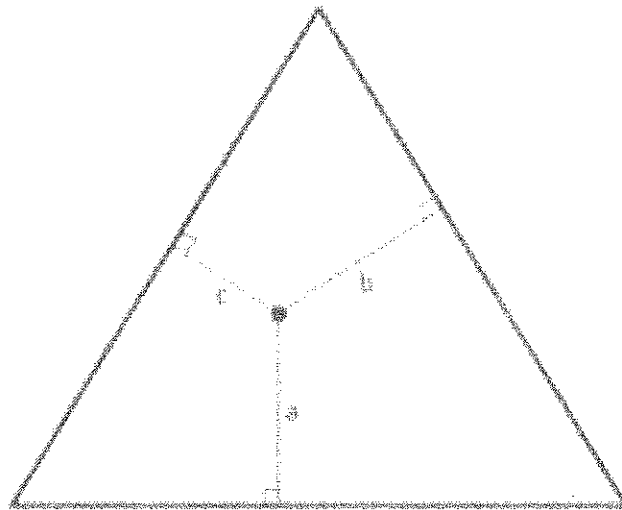
Scenario 1

We can now use what we just concluded and Viviani's Theorem to find the probability of obtaining three sticks that form a triangle in the first scenario:

- Nikola simultaneously breaks the stick of length 1, at random, into three pieces of lengths a , b , and c .

Notice that we are considering all possible values of a , b , and c where $a + b + c = 1$.

Problem 4. Consider an equilateral triangle of height 1. Using Viviani's theorem, explain why and how each point inside the equilateral triangle corresponds to breaking the stick into three parts.



$$a + b + c = 1$$

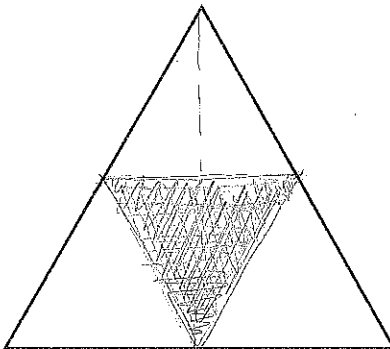
Each point in the triangle give us values of (a, b, c) where $a + b + c = 1$.

So each point represents a possible way the stick can be broken.

Problem 5. Of all the possible lengths a , b , and c can have, we have found earlier that a , b , and c must also satisfy

$$\begin{cases} a < \frac{1}{2} \\ b < \frac{1}{2} \\ c < \frac{1}{2} \end{cases}$$

For which points inside of the equilateral triangle will the above system of inequalities be satisfied? Shade in the area which correspond to these points on the triangle below. Assume that the triangle below has height 1.



Problem 6. Based on the picture you drew for Problem 4, if we were to simultaneously break a stick into three lengths randomly, what is the probability that the three resulting sticks will form a triangle?

Recall that

$$\Pr(\text{obtaining favorable region}) = \frac{\text{size of favorable region}}{\text{size of total region}}$$

$$\text{Probability} = \frac{1}{4}$$

Scenario 2

We will now find the probability of obtaining three sticks that form a triangle in the second scenario:

- Nikola breaks the stick of length 1, at random, into two pieces and then he breaks the right piece at random again.

Let a represent the length of the left piece of stick, and let b and c represent the lengths of the right pieces obtained this way.

Problem 7. Suppose Nikola is breaking the stick as described in the second scenario. (For this question, disregard whether or not a triangle can be formed.)

- (1) Nikola breaks the stick, at random, into two pieces. The piece on the left has length a . What are the possible values that a can be?

$[0, 1]$

- (2) Nikola then breaks the right piece at random. What are the possible values of b if we know what a is?

$[0, 1-a]$

- (3) Explain why possible values of b depend on a .

a determines how long the right stick is
and b is dependent on the length of
the right stick.

Problem 8. We say that Nikola broke the right piece of stick randomly because he randomly selected the proportion of the remaining stick to be the middle piece.

However, the actual length of the middle piece depends on how much was broken off in the first break.

Let b be the length of the second piece of the three pieces.

- (1) Find the ratio of the length of b to the length of the right piece after the first break in terms of a and b . Let this ratio be p .

$$p = \frac{b}{1-a}$$

- (2) Express b in terms of a and p .

$$b = p \cdot (1-a)$$

- (3) Express c in terms of a and p .

$$c = 1 - b - a = 1 - p(1-a) - a$$

- (4) What are the ranges of a and p ?

$$a \in [0, 1]$$

$$p \in [0, 1]$$

Problem 9. Since we are randomly choosing $a \in [0, 1]$ and $p \in [0, 1]$, these are the two variables we consider when finding geometric probability.

- (1) In the grid below, draw the area that represents all possible values that a and p can be. (Disregard whether or not a triangle can be formed.)
- (2) In the same grid, we will shade in the area that represents the values that a and p must be in order for the sticks to form a triangle.

Recall that in order for $a, b,$ and c to form a triangle, they must satisfy

$$\begin{cases} a < \frac{1}{2} \\ b < \frac{1}{2} \\ c < \frac{1}{2} \end{cases}$$

Rewrite the system of inequalities in terms of a and p by using your answers from parts (2) and (3) of Problem 7. Shade the area corresponding to values of a and p which satisfy the system of inequalities in the grid before.

(a) Shade in the area that satisfies $a < \frac{1}{2}$ in the grid below in yellow.

(b) Finding the area that satisfies $b < \frac{1}{2}$:

(i) Rewrite $b < \frac{1}{2}$ in terms of a and p .

$$p(1-a) < \frac{1}{2}$$

(ii) Simplify the inequality so that p is on the left hand side of the inequality and a is on the right hand side of the inequality.

$$p < \frac{\frac{1}{2}}{(1-a)} = \frac{1}{2(1-a)}$$

(iii) Plot the above inequality by finding the points in the graph for

$$a = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$$

$$(0, \frac{1}{2})$$

$$(\frac{1}{8}, \frac{4}{7}) \approx (\frac{1}{8}, 4.57/8)$$

$$(\frac{1}{4}, \frac{2}{3}) \approx (\frac{1}{4}, 5.33/8)$$

$$(\frac{3}{8}, \frac{4}{5}) \approx (\frac{3}{8}, 6.4/8)$$

$$(\frac{1}{2}, 1)$$

(iv) Shade in the area that satisfies the inequality in red.

(c) Finding the area that satisfies $c < \frac{1}{2}$:

(i) Rewrite $c < \frac{1}{2}$ in terms of a and p .

$$1 - p(1-a) - a < \frac{1}{2}$$

(ii) Simplify the inequality so that p is on the left hand side of the inequality and a is on the right hand side of the inequality.

$$p > 1 - \frac{1}{2(1-a)}$$

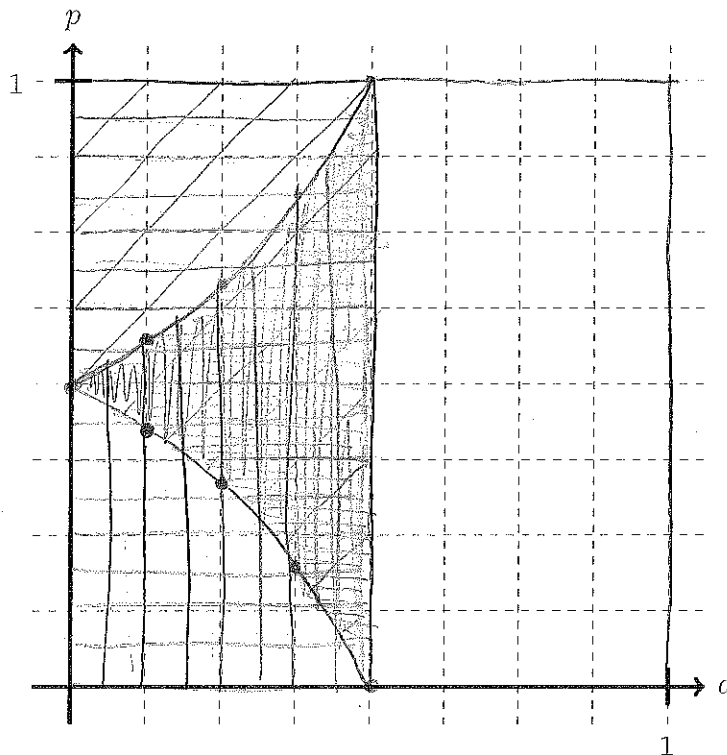
(iii) Plot the above inequality by finding the points in the graph for

$$a = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$$

$$\begin{aligned} &(0, \frac{1}{2}) \\ &(\frac{1}{8}, \frac{3}{4}) \\ &(\frac{1}{4}, \frac{1}{3}) \\ &(\frac{3}{8}, \frac{1}{5}) \\ &(\frac{1}{2}, 0) \end{aligned}$$

(iv) Shade in the area that satisfies the inequality in blue.

(d) Shade in the area where all three equations are satisfied in black.



Problem 10. Finding the probability.

- (1) Approximate the area shaded in the graph by counting squares.

Approximately 13 squares

- (2) Using your answer from part (1), approximate the likelihood that the three resulting sticks will form a triangle in the second scenario.

$$\frac{13 \text{ squares}}{64 \text{ squares}} \approx 0.203 = 20.3\%$$

- (3) Which scenario will more likely result in three sticks that form a triangle?

The first scenario.

Problem 11. Give an intuitive explanation as to why one scenario is more likely to happen.

Answers may vary.

We're more likely to end up with a break past the mid point of the stick (~50%) from the second scenario than the first, so the first scenario is more likely.