PLANAR GRAPHS

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OLGA RADKO MATH CIRCLE ADVANCED 2

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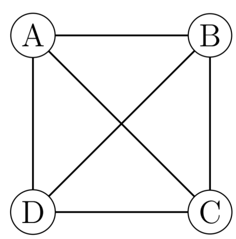
1 Introduction

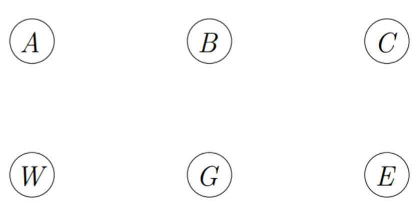
Recall that a graph G is a pair (V, E), where V is a set called the set of *vertices* and E is a set consisting of pairs of elements in V, called *edges*. We can draw graphs by drawing vertices as dots or circles, and edges by lines between them. For example, the graph with vertices $V = \{A, B, C, D\}$ and edges $E = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$ can be drawn as follows:



Note that the same graph can be drawn in many different ways. A graph is called *planar* if it can drawn in the plane in such a way that no edges cross one another.

Problem 1. Show that the above graph is planar.

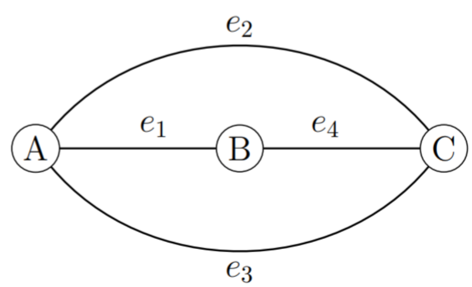
Problem 2. Do you think it is possible to connect three houses, A, B, and C, to three utility sources, water (W), gas (G), and electricity (E), without using the third dimension, so that the utility lines do not intersect?



Although you could probably give some intuitive reasons for why it was not possible to connect the houses in the previous problem, it's hard to argue precisely because there are so many different ways to try connecting the vertices. The goal of today's worksheet is to be able to really understand why this is impossible, and then we will characterize *all* planar graphs with a simple criterion.

2 Euler's characteristic formula

Before we get to the problem in the introduction, we need to take a small detour. Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior face* or *infinite face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1 bounded by the edges e_1, e_2, and e_4, and F_2 bounded by the edges e_1, e_3, and e_4. Its exterior face F_3 is bounded by the edges e_2 and e_3.

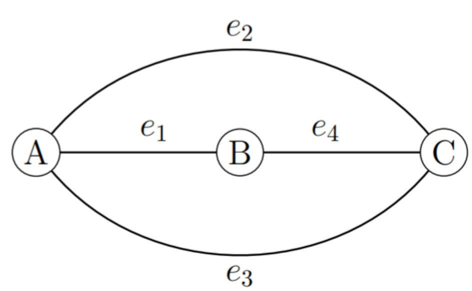


The *Euler characteristic* of a graph is the number of the graph's vertices minus the edges plus the number of the faces,

$$\chi = V - E + F.$$

Problem 3. Compute the Euler characteristic of the following graphs.

1.



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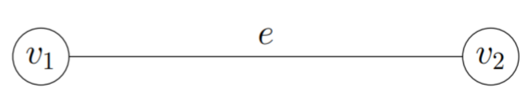
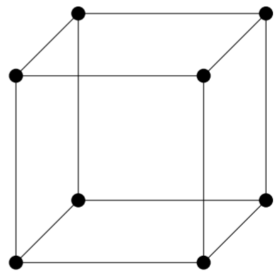
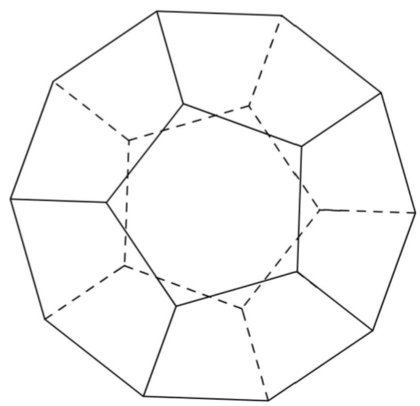


Problem 4. Show that each of the following graphs are planar by drawing them without intersecting edges, then compute the Euler characteristic of the graphs.

1. 

2. 

Can you make a conjecture about the Euler characteristic of every planar graph? Let us now prove the theorem.

Theorem 1 (Euler's characteristic formula)**.** Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.

Remember that connected just means that you can go from any vertex to any other vertex by travelling along edges. All of the graphs we have seen in this worksheet are connected.

Problem 5. Let us prove the theorem with the following steps.

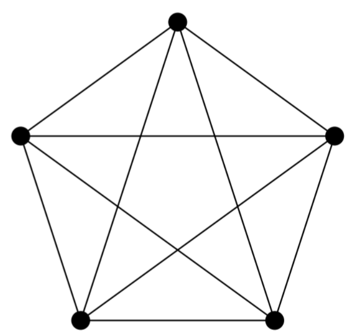
1. Recall that a graph is called a tree if it has no cycles (i.e. a way to go out of a vertex and come back without reusing edges). Explain why every tree is planar.
2. Explain why every tree has Euler characteristic 2. This proves Euler's formula in the case where the graph has no cycles.
3. Now assume that the graph has one cycle. Explain why Euler's formula is true in this case. (Hint: what happens when you break the cycle by removing an edge?)
4. Formalize the above argument with induction (base case and inductive step).

Solution.

1. Draw the tree as a rooted tree, then it will be planar.
2. There is one exterior face and no interior faces, and $E = V - 1$, so $V - E + F = V - (V - 1) + 1 = 2$.
3. When you break the cycle by removing an edge, the face inside the cycle and the exterior face are merged into one. (Because every cycle has two sides by the Jordan curve theorem, but they don't need to cite this.) Hence the number of faces decreases by 1 and the number of edges also decreases by 1, so the value of $V - E + F$ doesn't change.
4. By induction on the number of edges. A minimally connected graph is a tree, with $n-1$ edges, this is the base case. Then, any graph on $m > n-1$ edges has some cycle, so by identical logic to part 3, we can break the cycle and observe that the value of $V - E + F$ does not change. □

3 K_5 and $K_{3,3}$

Finally, with the tool of Euler's formula, let us return to the problem in the introduction. We wanted to develop a method to show that certain graphs are *not* planar. Let's start with a slightly different graph to make the method clear. This graph is called K_5.



We will also need one more definition. A graph is *simple* if there is at most one edge between every pair of vertices and no edges from any vertex to itself. All of the graphs we have considered so far have been simple, except the first graph in the previous section.

Problem 6.

1. Let a finite connected simple graph planar graph have $E > 1$ edges and F faces. Prove that then $2E \geq 3F$. (Hint: think about the number of edges on each face. If you add them all up, at most how many times did you count each edge?)
2. Using Euler's formula, prove that then $E \leq 3V - 6$.
3. Using the above fact, explain why K_5 is not planar.

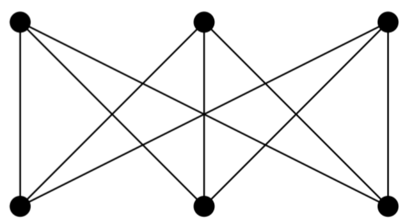
Solution.

1. There are at least 3 edges surrounding every interior face, as well as the exterior face unless there are less than 3 edges in total (treat those cases separately). Add up the number of edges on each face and call the sum x. By the argument above, $x \geq 3F$. At the same time, we counted each edge at most twice because every edge belongs to at most 2 faces. Hence, $x \leq 2E$. So $2E \geq x \geq 3F$.

2. Plug in $F = 2 - V + E$.

3. K_5 has 10 edges and 5 vertices, and $10 \not\leq 9$. But K_5 is finite, connected, and simple, so it must fail to be planar. □

Now, we are ready to do almost the same thing with the graph from the introduction! That graph is commonly called $K_{3,3}$.



Problem 7.

1. Let a finite connected simple graph planar graph have $E > 1$ edges and F faces. Suppose further that the graph has no cycles of length 3, i.e. triangles. Prove that then $E \geq 2F$.

2. Using Euler's formula, prove that then $E \leq 2V - 4$.

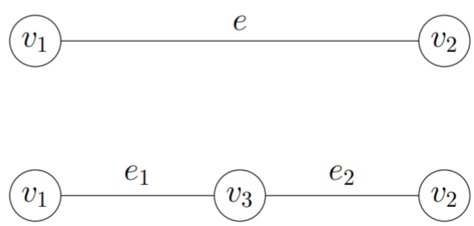
3. Using the above fact, explain why $K_{3,3}$ is not planar.

Solution. Exactly the same thing, now with at least 4 edges on every face. □

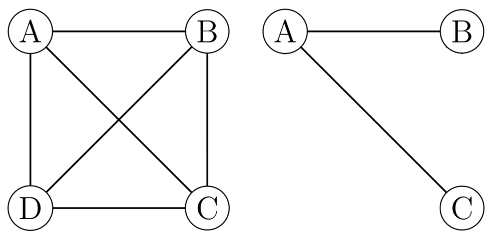
4 Kuratowski's theorem

Amazingly, the two graphs on the previous page are in some sense, "representative" of all non-planar graphs! In this section, we will understand what we mean by this in Kuratowski's theorem.

The *subdivision* of an edge $e = \{v_1, v_2\}$ is a graph containing one new vertex v_3, with the edges $e_1 = \{v_1, v_3\}$ and $e_2 = \{v_3, v_2\}$ replacing the edge e. A subdivision of a graph G is the result of starting with G. In the below example, the second graph is a subdivision of the first graph.



A graph H is called a *subgraph* of a grpah G if the sets of vertices and edges of H are subsets of the sets of vertices and edges of G. In the below example, the graph on the right is a subgraph of the graph on the left.



Theorem 2 (Kuratowski's theorem)**.** A graph G is not planar if and only if some subdivision of $K_{3,3}$ or K_5 is a subgraph of G.

Problem 8. Although the "only-if" direction is significantly more challenging to prove, we can prove the "if" direction quite easily. That is, we will show that if some subdivision of $K_{3,3}$ or K_5 is a subgraph of G, then G is not planar.

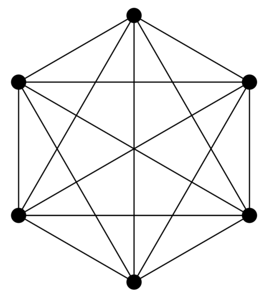
1. Explain why any subdivision of a non-planar graph is non-planar.
2. Explain why any subgraph of a planar graph is planar.
3. Conclude that if some subdivision of $K_{3,3}$ or K_5 is a subgraph of G, then G is not planar.

Solution.

1. If the subdivision was planar, then we could "un-subdivide" the graph by "erasing" the vertices and get a planar drawing of the original graph.
2. If we draw the original graph in a planar way and just erase things until we get the subgraph, there will not be any edge crossings.
3. Every subdivision of $K_{3,3}$ or K_5 is not planar by (1). The contrapositive of (2) is that if a graph contains a non-planar subgraph, then it is not planar. Hence, if a graph contains a subdivision of $K_{3,3}$ or K_5, then it is not planar. $\square$

Problem 9. Use Kuratowski's theorem to decide whether or not the following graphs are planar or not. That is, either draw the graph in a planar way or find a subdivision of K_5 or $K_{3,3}$ as a subgraph of the graph.

1.



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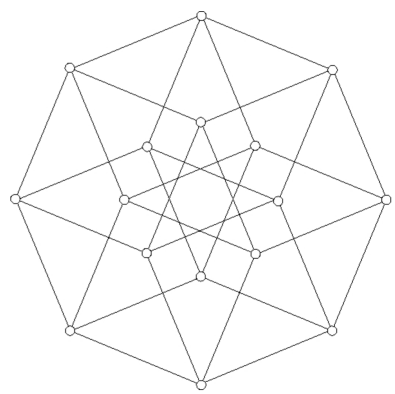


Solution.

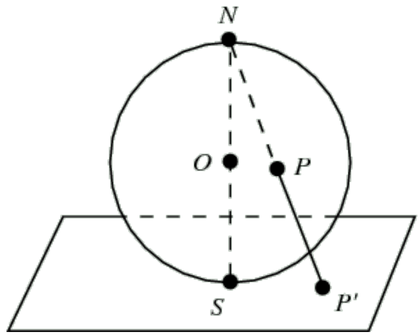
1. The graph contains K_5 as a subgraph quite obviously.
2. The graph contains a subdivision of $K_{3,3}$, although this may be quite difficult to find. It could be fun to try finding it with students.

   The graph also fails $E \leq 2V - 4$, which applies because it has no triangles. □

5 Challenge: Planarity on other surfaces

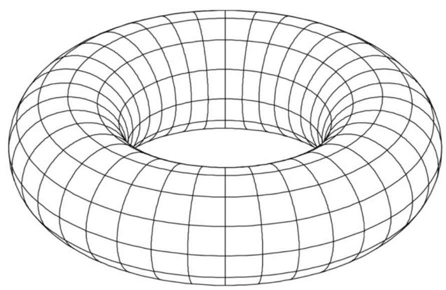
In this handout, we always considered graphs drawn in the plane. But being mathematicians, we are always interested in ways we can generalize ideas. For example, what if we drew graphs on a sphere? It turns out that not much changes, and in fact it motivates why we count the external infinite face of a planar graph.

Problem 10. Using the following picture, explain why a graph can be drawn in the plane without intersecting edges if and only if it can be drawn on a sphere without intersecting edges. This picture is called *stereographic projection*.

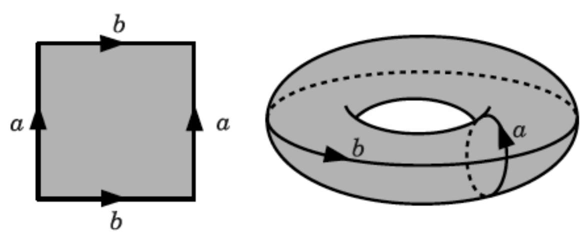


Solution. This is quite trivial but one pedantic thing to note is that you have to make sure no vertex is drawn at the north pole. □

Things get a little more interesting when we draw graphs on a torus instead. Recall that a torus is the following shape:

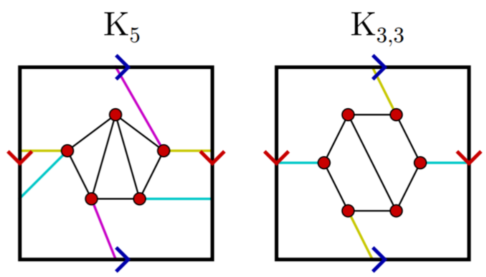


If you have difficulty thinking in 3 dimensions, it may be easier to think about the torus as a square with sides glued together, as in the game Pac Man but with the top and bottom glued as well.



Problem 11. Show that K_5 and $K_{3,3}$, although not planar, are both toroidal, meaning they can be drawn on a torus without intersecting edges.

Solution.



□

One thing to be aware of is that on the torus, some regions can be topologically different from others. For example, imagine drawing a small triangle graph on the torus. The triangle splits the torus into two regions, a small region on the inside and a large region on the outside. The small region on the inside feels similar to any face on a graph drawn in the plane. But the large region is very strange, because you can draw closed loops that cannot be shrunk down to a point! (Your lead instructor is too lazy to draw you a new picture here, but if you can imagine the triangle on the gray torus above, the loops labeled "a" and "b" both cannot be shrunk down to a point.)

In light of this observation, in order for a drawing of a graph on a torus to be valid, we require that any closed loop in any face can be continuously shrunk down to a point on the surface. In other words, every face must be what is called a *simply connected domain*. Check to make sure that in your drawings of K_5 and $K_{3,3}$, your faces indeed satisfied this property. With this understanding of what a face is, it is possible to generalize some results of planar graphs to toroidal graphs.

Problem 12. Try to show that connected toroidal graphs satisfy $V - E + F = 0$.

Solution. This is actually not so trivial because the base case of tree used in the proof for planar graphs is no longer a toroidal graph. You need to have edges that look "something like" a and b in the gray torus above in order to divide the torus into simply connected domains. But then it's not so clear how to start the induction.

A full proof can be found in these lecture notes. But this is more just something interesting for the students to think about. □