BORSUK-ULAM THEOREM AND APPLICATIONS

OLGA RADKO MATH CIRCLE
ADVANCED 3
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Crawl before you walk

Functions are fun, but they can be wild. Today we will explore how seemingly innocent constraints on our functions can lead to incredible consequences.

A function $f : X \to Y$ between sets X and Y is a map that associates to every element of X exactly one element of Y. Nothing more can be said about functions in general. For this reason, we often consider functions between sets with additional structure. The two families of sets we will care about are S^n and $\mathbb{R}^n$.

Definition 1. Let $\mathbb{R}^n = \{(a_1, ..., a_n) : a_i \in \mathbb{R}\}$ be the set of vectors with entries in the real numbers.

For example:

1. $\mathbb{R}^1 = \mathbb{R}$ is the real number line.
2. $\mathbb{R}^2$ can be thought of as the plane.
3. $\mathbb{R}^3$ can be thought of as the 3-dimensional space.

Definition 2. Let $S^n = \{(a_1, ..., a_{n+1}) \in \mathbb{R}^{n+1} : a_1^2 + ... + a_{n+1}^2 = 1\}$ be the set of points in $\mathbb{R}^{n+1}$ of distance 1 away from the origin.

Note: The n in S^n refers to the dimension of the sphere, not the dimension of the space that the sphere sits in (which is one greater).

Por ejemplo:

1. S^1 is the circle sitting in the plane.
2. S^2 is our typical sphere in 3-dimensional space.

Problem 1. What is S^0?

The last condition to discuss is what it means for a function to be *continuous*. For the sake of this topic, the following **informal** definition will suffice.

Definition 3 (Informal)**.** A function $f : S^n \to \mathbb{R}^n$ is *continuous* if any two arbitrarily close points in S^n get sent to arbitrarily close points in $\mathbb{R}^n$.

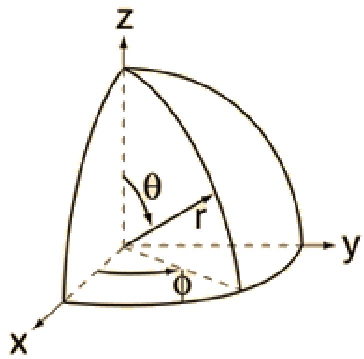
Problem 2. Let's explore continuous maps from S^1 to $\mathbb{R}$.

(a) Viewing S^1 as $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, give an example of a continuous function $f : S^1 \to \mathbb{R}$.

(b) Using polar coordinates, we can parameterize S^1 by an angle in $[0, 2\pi]$, with 2π representing the same point as 0 (the radius for each point is 1). Use this parametrization to give an example of a continuous function $f : [0, 2\pi] \to \mathbb{R}$. (You must make sure that $f(2\pi) = f(0)$ for this to be well defined)

(c) For each of your examples, do there exist antipodal points (points opposite each other) on the circle with the same output? If so, find them. If not, prove it!

Problem 3. We will now repeat the previous problem but for S^2. Recall the spherical coordinates on S^2 where $\theta \in [0, \pi]$ is the angle downwards from the north pole and $\phi \in [0, 2\pi)$ is the angle along the equator in the xy-plane starting from the positive x-axis. For example, the point $(0, 0, 1)$ would have $\theta = 0$ and $(0, 0, -1)$ would have $\theta = \pi$. See the figure below ($r = 1$ on the unit sphere):



(a) Viewing S^2 as $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, give an example of a continuous function $f : S^2 \to \mathbb{R}^2$. (Hint: The restriction of any continuous function is continuous)

(b) Viewing S^2 in spherical coordinates, give an example of a continuous function

$$f : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^2.$$

What does it mean to be well defined in this case?

(c) For each of your examples, do there exist antipodal points (points opposite each other) on the sphere with the same output? If so, find them. If not, prove it!

Problem 4. Make a conjecture about continuous maps from S^n to $\mathbb{R}^n$ using the previous two problems as motivation.

Unfortunately, you were about 90 years too late with your conjecture! What can you do, math is like that sometimes.

Theorem 1 (**Borsuk-Ulam Theorem**). *If $f : S^n \to \mathbb{R}^n$ is continuous, then there exists an $x \in S^n$ such that $f(x) = f(-x)$. In words, there are antipodal points on the sphere whose outputs are the same.*

Problem 5. Consider the Borsuk-Ulam Theorem above.

(a) What restrictions are you putting on the set of all functions? In other words, what choices are you making?
(b) What does the theorem reward you with for making those choices?

Problem 6. (Earth, Wind, and Fire)

Prove that at any time of day, there exist locations on the surface of the Earth directly opposite each other with the exact same wind speed and temperature. Make sure to specify what assumptions you are making and why they are reasonable assumptions to make. (Whenever we apply math to the real world, we need to give justification.)

Proof of Borsuk-Ulam when $n = 1$

In order to prove the 1-dimensional case of the Borsuk-Ulam theorem, we must recall a theorem about continuous functions which you have probably seen before.

Theorem 2 (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on an interval $[a, b] \subset \mathbb{R}$. Moreover, suppose that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = 0$.

Note: This theorem is actually very intuitive. It's saying that if you start on one side of zero and end on another side of 0, and you are continuous, then you must have passed zero at some point.

Let's see how we can use the intermediate value theorem to prove the Borsuk-Ulam theorem for $n = 1$.

Problem 7. Let $f : S^1 \to \mathbb{R}$ be a continuous function from the circle to the real numbers. In problem 2, we saw that we can rewrite f as a continuous function $f : [0, 2\pi] \to \mathbb{R}$ with the extra property that $f(0) = f(2\pi)$. Our goal is to find a pair of antipodal points with the same value under f.

(a) Define a function g which outputs the difference between f evaluated at a point on the circle and f evaluated at its antipodal point. Make sure to specify the domain of g. Is this function continuous?

(b) Use part (a) and the Intermediate Value Theorem to prove that there exist antipodal points a, b such that $f(a) = f(b)$.

Unfortunately, the higher dimensional cases of the Borsuk-Ulam theorem require a bit more machinery to prove. So at this point in time, we will take Borsuk's word for it and believe that the theorem is true in all dimensions.

Corollaries and Consequences

Eight out of nine dentists agree that the best part of the Borsuk-Ulam theorem is all the fun stuff you can prove using it. The ninth dentist never responded to the poll, but researchers believe the ninth's opinion would not deviate from the herd.

Why save the best for last when you can not?

Theorem 3 (**Brouwer's fixed point theorem**)**.** If $f : B^n \to B^n$ is a continuous map from the n-ball to itself, then there exists a point $x \in B^n$ such that $f(x) = x$.

We call such a point a **fixed point** of f because it remains unchanged under the map.

Recall: The n-ball, B^n, is the region bounded by S^{n-1}. This is the set of solutions to $x_1^2 + ... + x_n^2 \leq 1$.

You can think of the condition that f must be continuous as saying you can stretch, shrink, and fold the ball in any direction so that it still fits in the original ball but you cannot tear any part of it.

We must first prove two lemmas before we can tackle this one.

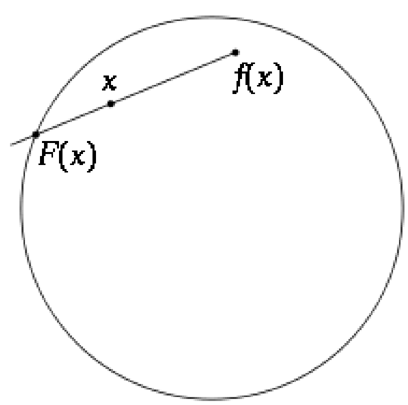
Problem 8. Show that there is no continuous map $f : S^n \to S^{n-1}$ satisfying $f(x) = -f(-x)$ for all $x \in S^n$. (Hint: Use the Borsuk-Ulam theorem.)

Definition 4. A function satisfying $f(x) = -f(-x)$ for all x in its domain is called an **odd** function.

Problem 9. Show that there is no continuous map $f : B^n \to S^{n-1}$ which is odd when restricted to its boundary S^{n-1}. (Hint: Suppose there exists such an $f : B^n \to S^{n-1}$. Define $g : S^n \to S^{n-1}$ by $g(x_1, ..., x_{n+1}) = f(x_1, ..., x_n)$.)

We are now ready to prove the celebrated **Brouwer's fixed point theorem**, which you may recall from a previous worksheet.

Problem 10. Suppose, to the contrary, that there is a function $f : B^n \to B^n$ with no fixed points. Define a function $F : B^n \to S^{n-1}$ that does the following:
Let $x, f(x) \in B^n$. Draw a ray starting at $f(x)$ and going through x. Now follow the ray until you hit the boundary of B^n, which is S^{n-1}. The point where the ray hits the boundary will be $F(x)$. This is illustrated in the picture below for the case $n = 2$.



(a) Prove that this map is well defined. That is, how do we know that every input has a unique output?

(b) Give an informal explanation for why F is continuous, i.e. if $x, y \in B^n$ are close together, why are $F(x), F(y) \in S^{n-1}$ close together?

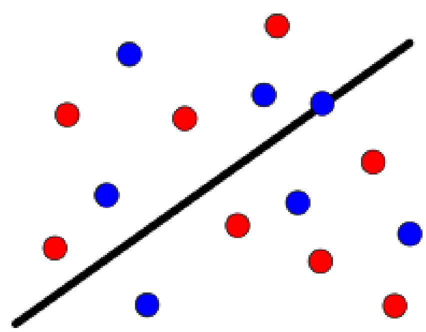
(c) Use problem 9 to arrive at a contradiction.

Brouwer's fixed point theorem has many surprising applications which we might get to see in a future handout.

Our next theorem which follows from Borsuk-Ulam has the second best name out of any theorem in topology. More importantly, this theorem is incredibly useful if you are super hungry late at night.

Theorem 4 (**Ham Sandwich**)**.** Let $W_1, W_2, ..., W_d$ be nice regions in $\mathbb{R}^d$ with nonzero volume. Then there is a hyperplane that simultaneously bisects each W_i.

In this context, a nice region refers to a bounded region which has finite volume and finitely many connected components (think of physical objects). A hyperplane is a generalization of a plane in $\mathbb{R}^3$, so it is an $d-1$ dimensional slice of $\mathbb{R}^d$. Notice that a hyperplane splits $\mathbb{R}^d$ into two regions, one on either side of the hyperplane. We say that a hyperplane bisects a set W if it splits W into equal volume parts on either side of the hyperplane.



The 2-dimensional case of the Ham Sandwich theorem, where W_1 is the union of the blue circles, and W_2 is the union of the red circles.

Let's look at the case $n = 3$, which will give some insight into the naming of the theorem. The theorem says that given three regions W_1, W_2, W_3 of $\mathbb{R}^3$, there exists a plane which cuts each region into equal parts simultaneously. Let's imagine that W_1 is a piece of bread, W_2 is a piece of ham, and W_3 is a piece of bread. The ham sandwich theorem says that no matter how you make your sandwich, given an infinitely long knife, you can make a single straight cut and split the sandwich in half so that each half has the same amount of bread and ham as the other half. Notice that the theorem doesn't require the original pieces of bread and ham to be connected (for example, the ham could start out split into thousands of slices spread across the world).

We will prove the case for $n = 3$, from which the general case follows in a similar manner.

First, we have to find the right way to think about these hyperplanes. In $\mathbb{R}^3$, these are actual planes.

To each plane through the origin, we can associate a **normal vector** in $\mathbb{R}^3$. This vector points perpendicular to the plane. If we only care about the direction of the normal vector, then we can choose the normal vector to be a point on the sphere, S^2. Each choice of normal vector specifies a positive region of $\mathbb{R}^3$, since the plane divides $\mathbb{R}^3$ into two regions and the normal vector sits in one of these regions. Call the region which the normal vector points into the positive region.

Problem 11. Draw a picture of a hyperplane in $\mathbb{R}^3$. Make sure to include a normal vector and label the positive region with respect to your choice of hyperplane and normal vector.

Problem 12. If we have a plane not through the origin, what do we have to specify in addition to the normal vector in order to fully describe it?

Let W_1, W_2, W_3 be our nice regions in $\mathbb{R}^3$. We want to find a plane which simultaneously bisects all three regions.

Problem 13. Choose a normal vector $v \in S^2$ and a real number $h \in \mathbb{R}$. Let $P_{v,h}$ be the plane which has normal vector v and point $hv \in P_{v,h}$. Let $f_{i,v,h}$ be the volume of W_i which is on the positive half of $P_{v,h}$.

(a) Use the intermediate value theorem to prove that for each $v \in S^2$ there exists an h' such that $P_{v,h'}$ bisects W_1. Denote this plane by P_v.

Let $f_{i,v}$ be the volume of W_i on the positive half of P_v. Define $f : S^2 \to \mathbb{R}^2$ to be the function that sends $v \to (f_{2,v}, f_{3,v})$, where $f_{2,v}$ and $f_{3,v}$ are the volumes of W_2 and W_3 on the positive half of P_v.

(b) Give an informal justification for why we can expect f to be a continuous function.

(c) If $P_{v,h'}$ bisects W_1, what value of h should you choose so that $P_{-v,h} = P_{-v}$ is the same plane as $P_{v,h'}$? What is the difference between P_v and P_{-v}?

(d) Show that $f_{i,v} + f_{i,-v} = Volume(W_i)$.

(e) Use the Borsuk-Ulam theorem to prove the Ham Sandwich theorem in dimension 3.

(f) How would you extend this proof to higher dimensions?

Problem 14. (**Challenge**: Throwback to last quarter's real projective spaces)
Recall that the real projective space of dimension n, $\mathbb{RP}^n$ is the manifold given by the set of all lines in $\mathbb{R}^{n+1}$ which pass through the origin. We've seen a few descriptions for this manifold.

(a) Prove that $\mathbb{RP}^n$ can be thought of as the set of all hyperplanes through the origin.

(b) Give a bijection between the set of hyperplanes through the origin and pairs of antipodal points on S^n.

(c) Using the notation from the previous problem, let $F : S^2 \to \mathbb{R}^2$ be given by

$$F(v) = (f_{2,v} - f_{2,-v}, f_{3,v} - f_{3,-v}).$$

Prove that F is an odd function.

(d) Show that if the Ham Sandwich theorem were not true, then F would induce a map

$$\tilde{F} : \mathbb{RP}^2 \to \mathbb{RP}^1.$$

(Hint: If the image of F avoids $(0,0)$, show that you can rescale F to give a map $S^2 \to S^2$. Now you must show that each antipodal pair gets sent to an antipodal pair, which will lead to a well-defined map on projective spaces.)

Our last theorem has an outstanding international prize of 527 Yen for whoever can correctly pronounce its name. Let's first see what we think in lower dimensions before tackling the general theorem.

Problem 15. (a) Can you cover the circle with two arcs such that neither arc contains a pair of antipodal points? How about three arcs? Prove it!

(b) Cover the sphere with four connected regions so that none of the regions contain a pair of antipodal points. Can you still do this with only three regions?

(c) Make a conjecture about how many regions you need in order to cover the n-dimensional sphere S^n so that none of them contains a pair of antipodal points.

Now that we got our hands dirty in dimensions 1 and 2, we are ready to generalize.

Theorem 5 (Lusternik–Schnirelmann theorem). If the sphere S^n is covered by $n+1$ sets (not necessarily connected), then one of the sets contains a pair $(x, -x)$ of antipodal points.

Note: You can either take all of the sets to be open (not containing their boundary) or all to be closed (containing their boundary).

Problem 16. Suppose that the sets $U_1, U_2, ..., U_{n+1}$ cover S^n. Define $d(x, U_i)$ to be the distance from $x \in S^n$ to the nearest point on U_i.

(a) If $x \in U_i$, what is $d(x, U_i)$?

(b) Define $f : S^n \to \mathbb{R}^n$ by $x \mapsto (d(x, U_1), d(x, U_2), ..., d(x, U_n))$. Apply Borsuk-Ulam to reduce Lusternik-Schnirelmann to two cases and prove it!

(c) Is it always possible to cover S^n with $n+2$ sets such that no set contains a pair of antipodal points? Think geometrically.

Problem 17. (Manifolds)
Recall that two topological spaces X and Y are *homeomorphic* if there exists a continuous bijection $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$.
Prove that S^n and $\mathbb{R}^n$ are **not** homeomorphic.

Equivalences of our many results

We have proved quite a bit of theorems, lemmas, and corollaries so far. But maybe they all imply each other and are therefore equivalent!

Problem 18. Prove that the following are equivalent:

(1) Borsuk-Ulam Theorem

(2) Every continuous odd function $f : S^n \to \mathbb{R}^n$ has a point $x \in S^n$ such that $f(x) = 0$.

(3) There does not exist a continuous odd function $f : S^n \to S^{n-1}$.

(4) A covering of S^n by $n+1$ open (or closed) sets has at least one set containing a pair of antipodal points.