ORMC: INEQUALITIES

OLYMPIAD GROUP 1, WEEK 2

Problem 1. (Textbook examples)

(a) Show that $a^2 + b^2 \geqslant 2ab$ for all real numbers $a, b \in \mathbb{R}$. When do we get equality?

(b) Show that $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ for all real numbers $a, b, c \in \mathbb{R}$. When do we get equality?

(c) If $a, b \in \mathbb{R}$ are fixed real numbers, then show that $x^2 + ax + b$ reaches its minimal value at $x = -\frac{a}{2}$. What is this value?

Problem 2. Let $x, y > 0$ be real numbers and $n > 1$ an integer. Show that $x^n + y^n \geqslant x^{n-1}y + xy^{n-1}$. When do we get equality?

Problem 3.

(a) Show that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

(b) Conclude that if $a, b, c > 0$, then $a^3 + b^3 + c^3 \geqslant 3abc$. When do we get equality?

[*AM-GM Inequality*]

Problem *4. Find the minimum of $5x^2 + 2xy + 2y^2 - 14x - 10y + 17$ where x, y are arbitrary real numbers. *Hint: Use Problem 1 (c).*

Problem 5.

(a) Let a, b, x, y be positive real numbers. Show that

$$\frac{a^2}{x} + \frac{b^2}{y} \geqslant \frac{(a+b)^2}{x+y}$$

(b) More generally, show that

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geqslant \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \ldots + x_n}$$

for all $a_1, \ldots, a_n, x_1, \ldots, x_n$ positive real numbers.

Problem 6. Paul wants to find the average number of people per household in a village. He proceeds by asking every single person how many people live in their household, and averaging all of the answers.

(a) Show that Paul's method is incorrect.

(b) How does Paul's answer compare to the real answer (i.e. which one is larger)?

Problem *7. Let

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2020^2 - 1} \qquad \text{and} \qquad B = \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2021}.$$

Show that $A > 2B$.

Homework

Problem 1. Find all positive integer solutions to the equation $m^2 + n^2 = mn + 7$.
Hint: You can use an inequality!

Problem 2. Let a, b, c be real numbers. Show that

$$a^4 + b^4 + c^4 \geqslant abc(a + b + c).$$

When do we get equality? *Hint: Try using Problem 1 (b).*