Olga Radko Endowed Math Circle

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The Number Line

Problem 1 *Prove that*

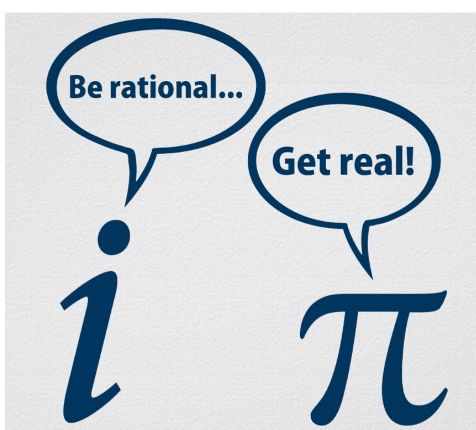
$$\frac{\frac{a}{b}}{c} = \frac{a}{bc} \tag{1}$$

for $b \neq 0$ and $c \neq 0$.

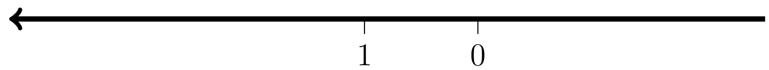
Definition 1 *A real number is a point on the number line.*

A <u>rational number</u> is a point on the number line that can be represented as a fraction m/n where m is an integer, n is a positive integer, and m and n have no common factors except for ± 1.

A point on the number line that cannot be represented as described above is an <u>irrational number</u>.

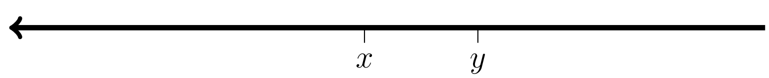


To introduce the structure of the number line on a straight line, we need to choose two distinct points, 0 and 1. We are free to choose the point 1 either to the left or to the right of the point 0. The choice gives the number line an *orientation*. Numbers increase in the direction from 0 to 1. This direction is called *positive*. Numbers decrease in the opposite direction. This direction is called *negative*. It is traditional to have the point 1 to the right of the point 0 on the number line. To emphasise that this actually is a matter of choice, let us take the point 1 to the left of the point 0 on the picture below.

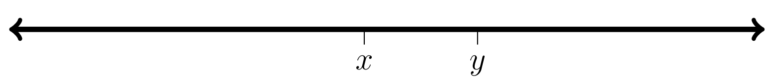


The arrow on the number line points in the positive direction. BTW, in most of the modern day US textbooks and workbooks, they draw arrows on both sides of the number line.

Doing this erases useful information. The number line is oriented: for any two distinct numbers x and y, either $x > y$ or $x < y$. Looking at a picture with one arrow, like the one below,

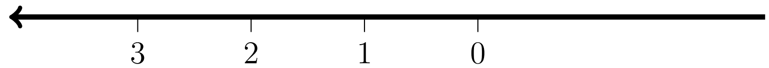


we know that $x > y$. Looking at a picture with two arrows,



we have no clue.

The distance between 0 and 1 on the number line is the measuring unit for constructing the rest of the numbers. If we take a unit step from 1 in the positive direction, we get 2. Taking one more step brings us to 3, and so on.



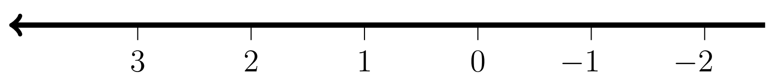
This way, we get the set $\mathbb{N}$ of *natural numbers.*

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

It is a recent tendency to include zero in the set. This is very wrong for historical reasons. The highly non-trivial idea to use a special symbol for *nothing* occurred to humanity tens, if not hundreds, of thousands of years after people had learned to use natural numbers for counting. Zero is too complicated to be natural! For example, we can divide any number by any number as long as the last is not equal to zero.

Problem 2 *Why is it not possible to divide by zero?*

Taking a unit step from 0 in the negative direction, we get -1. One more step brings us to -2, etc.



We get the set $\mathbb{Z}$ of *integral numbers*, or just *integers.*[1]

$$\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}$$

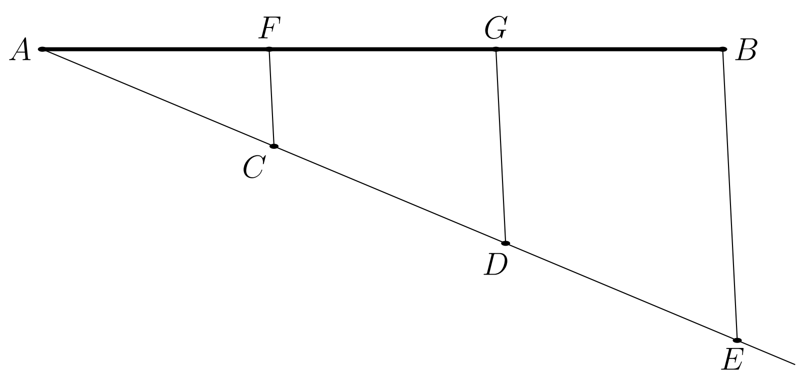
Note that any integer n can be constructed with a compass (geometric, not magnetic). One only needs to measure the unit length with the gadget and then to mark the proper number of steps in the right direction.

[1] Z is the first letter of *zahlen*, German for *numbers.*

Let n be a natural number. One can use a compass and straightedge to divide a straight line segment into n parts of equal length. Below, we divide the segment AB into three parts. The algorithm works the same way for any positive integer.

Given a straight line segment AB, let us draw a ray starting at A and not collinear with AB. Let us take any point C different from A on the ray. Let us further take points D and E on the ray so that the segments CD and DE have the same length as the segment AC.

Let us connect points B and E by a straight line segment. Let us further construct straight lines parallel to BE and passing through the points C and D. The points F and G where the lines meet the line AB will divide the segment AB into three parts of equal length. Take n equidistant points on the ray and the algorithm will divide AB into n equal parts.



Problem 3 *Prove that the segments AF, FG, and GB have equal length.*

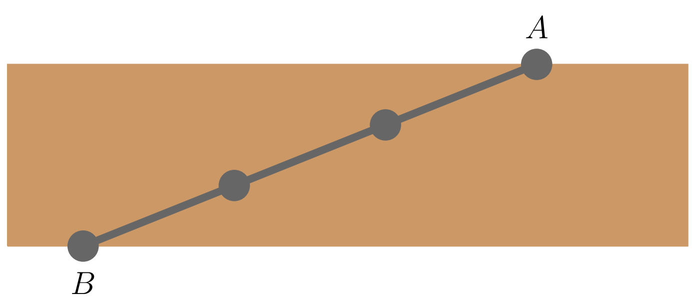
Now we can construct the set of rational numbers $\mathbb{Q}$ geometrically.[2] Take an integer M and a natural number N. Cancel out common factors, reducing M to m and N to n. Use the above algorithm to divide the segment from 0 to m into n equal parts. The closest point to 0 will be the rational number m/n. All the points of the number line that can be constructed this way are rational numbers. Others are not.

Most mathematical discoveries were made out of practical necessity. Imagine that you have to cut a slab into three oblong parts of equal width. The only measuring device you have is a rope. Measuring the width of the slab with the rope and then folding the rope into three parts of equal length does not work well because the rope is too thick. The following modification of the algorithm presented on page 4 solves the problem.

1. Make two knots, A and B, more than the slab's width, but less than the length of the slab's diagonal apart.

2. Fold the stretch of the rope between the knots into three parts of equal length, marked by two more knots.

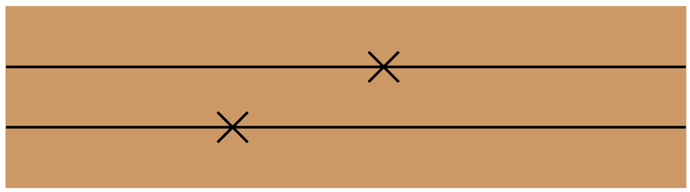
3. Position the rope along the slab as on the picture below.



4. Mark the positions of the two intermediate knots.

[2] Q is the first letter of the word *quoziente*, Italian for *quotient*. The symbol was introduced circa 1895 by a famous Italian mathematician, Giuseppe Peano.

5. Use the rope as a straightedge to draw the lines parallel to the longer sides of the slab through the marks.



6. Cut the slab along the lines.

Note that the knots A and B can be positioned anywhere on the opposite longer sides of the slab. As soon as the rope is stretched straight, the method will work. One can only agree with Kurt Lewin, 1890 - 1947, a German-American scientist known as the founder of social psychology, who is believed to coin the phrase "There is nothing more practical than a good theory!"

The symbol $\subseteq$ means a *subset* in the math language. For example, instead of writing that the set of natural numbers is a subset of the set of integers, we can write $\mathbb{N} \subseteq \mathbb{Z}$.

Recall that the set of all the points on the number line is called the set of real numbers. This set is denoted as $\mathbb{R}$.

Note that the number sets we have constructed are nested like a Russian matryoshka doll.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Problem 4 *Prove that $\sqrt{2}$ is not rational.*

Problem 5 *Use a compass and a ruler to locate the point $\sqrt{2}$ on the number line.*

Problem 6 *Prove that* $\sqrt{3}+\sqrt{2}$ *is not rational.*

Problem 7 *Find a pair of irrational numbers having a rational sum.*

Theorem 1 *A decimal number is rational if and only if it is either finite or recurring infinite.*

Recall that rational numbers are some special points on the number line. Definition 1 gives one way to represent a rational number. Decimals give another. For example,

$$\frac{1}{5} = 0.2 \quad \text{while} \quad \frac{1}{3} = 0.333\ldots = 0.\overline{3}.$$

Theorem 1 explains how the two ways are related. To prove Theorem 1, one has to understand a general recipe for going back and forth between the two representations.

Problem 8 *Prove Theorem 1.*

Problem 9 *Represent the number* $0.\overline{123}$ *as a fraction in lowest terms.*

Problem 10 *Construct an irrational number in a way that does not include taking a root of a rational number.*

Solving Problem 4, we have proven that $\sqrt{2}$ is not rational. Theorem 1 tells us that the decimal representation of $\sqrt{2}$ is therefore an infinite decimal without a recurring pattern. A calculator knows that $\sqrt{2} = 1.414213562\ldots$ How does it know? Is this correct? What if one needs more digits than the calculator can provide? It should be a bit scary to use a number as complicated as $\sqrt{2}$ without having answers to these questions.

Problem 11 *Find the first three digits of $\sqrt{2}$ after the decimal point without a calculator.*

Homework Problem 1 *Write a computer program that for any non-negative real number x finds $\sqrt{x}$ with any given precision $\epsilon > 0$.*

Problem 12 *Use a compass and ruler to locate the number* $\sqrt{3}$ *on the number line.*

Problem 13 *Check if the following equalities are correct.*

- $$\frac{3}{\sqrt{5}-\sqrt{2}} + \frac{5}{\sqrt{7}+\sqrt{2}} = \frac{2}{\sqrt{7}-\sqrt{5}}$$

The problem continues to the next page.

- $\dfrac{\sqrt{2}-1}{\sqrt{2}+1} = \sqrt[3]{\dfrac{10-7\sqrt{2}}{10+7\sqrt{2}}}$

Problem 14 *Simplify the following expression as much as possible.*

$$\left(\frac{\sqrt[4]{8}+2}{\sqrt[4]{2}+\sqrt[3]{2}} - \sqrt[3]{4}\right) \div \left(\frac{\sqrt[4]{8}-2}{\sqrt[4]{2}-\sqrt[3]{2}} - 3\sqrt[12]{128}\right)^{\frac{1}{2}}$$

Self-test questions

1. What is the number line?

2. What is a real number?

3. What is a rational number?

4. What is an irrational number?

5. Is the sum of irrational numbers always irrational? Why or why not?

6. What famous theorem helps one locate the number $\sqrt{2}$ on the number line?

7. What decimal numbers are rational and what are not?