LET'S GET AFTER IT!

OLGA RADKO MATH CIRCLE
ADVANCED 3
JANUARY 10, 2021

1. Hats, Bats, Rats, Hats

Problem 1. (**10 points**) Three people are each given a red or a blue hat at random. Each one can see the other people's hats, but not their own. They are told to raise their hands if they see someone wearing a red hat, and a prize is offered to the first person to (correctly) guess their hat color.
All three raise their hands, and several minutes pass. Then somebody guesses "My hat is red", winning the prize.
How did they know, and what colors were the other hats?

Solution. Suppose someone had a blue hat. Then all three would raise their hands. But the two others would see the blue hat and therefore know that their hat is red. Now if two people had a blue hat, the third wouldn't raise their hand. Therefore, the only option is for them to all have red hats.

The following are interactive hat problems. To play one of these problems, please read the problem carefully, agree on a strategy, and then gather a group, and head over to the main breakout room to play.

Problem 2. (**20 points**) Please gather a group of at least 6.
The instructor will assign each of you a hat color, red, green, or blue, at random. IRL, we might put a real hat on your head with your eyes closed, but today, the instructor will message everyone except you your hat color in the chat.
Once everybody is ready, then there are two rounds of guessing. In the first round, the instructor randomly chooses one of you, who guesses their hat color, which is allowed to be incorrect, but they are not told the answer. After two minutes of thinking independently, in the second round, everyone except that first representative must simultaneously guess their hat colors. This second time, you must all get it right.

Solution. Assign the different colors different numbers mod 3. Then the first person sums up all the hats mod 3 and guesses that color. Everyone else can find their color by subtracting the rest of the hats mod 3.

Problem 3. (**20 points**) Please gather a group of 3 people.
Each of you will be assigned a red or blue hat, and as before, you will only be told everybody else's hat color.
There is only one round of guessing, you all must guess your hat color, or pass, at once. You win as a group if not everyone passes, and everyone who doesn't pass guesses their hat color correctly.
Because this game is faster, you can play up to 4 times, and you get 5 point for each win

Solution. The hats should be as follows for both teams, to make it fair, as this game has no 100% winning strategy.

(1) RBR

(2) RBB
(3) BBB
(4) BRB

An optimal strategy is to pass whenever you see two hats of a different color, and whenever you see two hats of the same color, to guess the opposite color of that. This wins whenever the hats are not always the same color, which is an optimal 3/4 of the time.

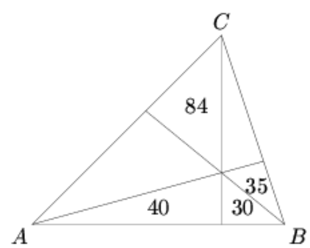
2. Gee, I'm a tree!

Problem 4. (**10 points**) Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

Solution. Let the radius of the circle be r, and let its center be O. Since $\overline{AB}$ and $\overline{AC}$ are tangent to circle O, then $\angle OBA = \angle OCA = 90°$, so $\angle BOC = 120°$. Therefore, since $\overline{OB}$ and $\overline{OC}$ are equal to r, then (pick your favorite method) $\overline{BC} = r\sqrt{3}$. The area of the equilateral triangle is $\frac{(r\sqrt{3})^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}$, and the area of the sector we are subtracting from it is $\frac{1}{3}\pi r^2 - \frac{1}{2}r \cdot r \cdot \frac{\sqrt{3}}{2} = \frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}$. The area outside of the circle is $\frac{3r^2\sqrt{3}}{4} - \left(\frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}\right) = r^2\sqrt{3} - \frac{\pi r^2}{3}$. Therefore, the answer is

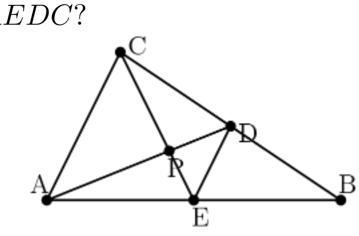
$$\frac{r^2\sqrt{3} - \frac{\pi r^2}{3}}{\frac{3r^2\sqrt{3}}{4}} = \boxed{\textbf{(E)}\ \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}}$$

Problem 5. (**10 points**) Triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



Solution. Using point masses, give B a weight of 4, A a weight of 3 and call the left unknown area S and the right unknown area T. Let x be the weight of C. Then we know that $x * T = 4 * 35$, $x * 84 = 3 * S$, and $x * (84 + S + T) = 4 * (40 + 30 + 35)$. Lastly, call h_1 the length of the line segment from the interior point down to AB, and h_2 the length of the line segment from C to the interior point. Then we have $7h_1 = x * h_2$. Therefore, $h_2 + h_1 = h_1(1 + \frac{7}{x})$. So the overall area is $(1 + \frac{7}{x}) * 70$ (the area of the triangle A,B, interior point). Solving our first three equations for x, we get $x = 2$, and so our total area is $70 * \frac{9}{2} = \mathbf{315}$.

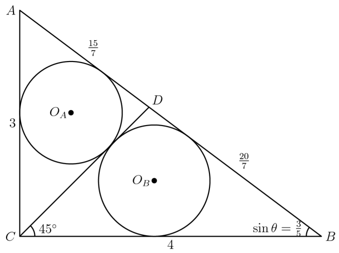
Problem 6. (**10 points**) In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



Solution. 13.5, mass points gives $CP = 3$, $PA = 4$. Then notice it's a right angle and work your magic.

Problem 7. (**10 points**) Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

Solution. (2008 AMC 12A Problem 20)



By the Angle Bisector Theorem,

$$\frac{BD}{4} = \frac{5 - BD}{3} \Longrightarrow BD = \frac{20}{7}$$

By Law of Sines on $\triangle BCD$,

$$\frac{BD}{\sin 45^\circ} = \frac{CD}{\sin \angle B} \Longrightarrow \frac{20/7}{\sqrt{2}/2} = \frac{CD}{3/5} \Longrightarrow CD = \frac{12\sqrt{2}}{7}$$

Since the area of a triangle satisfies $[\triangle] = rs$, where $r =$ the inradius and $s =$ the semiperimeter, we have

$$\frac{r_A}{r_B} = \frac{[ACD] \cdot s_B}{[BCD] \cdot s_A}$$

Since $\triangle ACD$ and $\triangle BCD$ share the altitude (to $\overline{AB}$), their areas are the ratio of their bases, or

$$\frac{[ACD]}{[BCD]} = \frac{AD}{BD} = \frac{3}{4}$$

The semiperimeters are $s_A = \left(3 + \frac{15}{7} + \frac{12\sqrt{2}}{7}\right) / 2 = \frac{18+6\sqrt{2}}{7}$ and $s_B = \frac{24+6\sqrt{2}}{7}$. Thus,

$$\begin{aligned}\frac{r_A}{r_B} = \frac{[ACD] \cdot s_B}{[BCD] \cdot s_A} &= \frac{3}{4} \cdot \frac{(24 + 6\sqrt{2})/7}{(18 + 6\sqrt{2})/7} \\ &= \frac{3(4 + \sqrt{2})}{4(3 + \sqrt{2})} \cdot \left(\frac{3 - \sqrt{2}}{3 - \sqrt{2}}\right) = \frac{3}{28}(10 - \sqrt{2})\end{aligned}$$

Problem 8. (**10 points**) Triangle ABC has $\angle C = 60^\circ$ and $BC = 4$. Point D is the midpoint of BC. What is the largest possible value of $\tan \angle BAD$?

Solution. (2008 AMC 12A Problem 24) We notice that $\tan(x)$ is strictly increasing on the interval $[0, \frac{\pi}{2})$ (if $\angle BAD \geq 90^\circ$, then it is impossible for $\angle C = 60^\circ$), so we want to maximize $\angle BAD$.

Consider the circumcircle of BAD and let it meet AC again at F. Any point P between A and F on line AC is inside this circle, so it follows that $\angle BPD > \angle BAD$. Therefore to maximize $\angle BAD$, the circumcircle of BAD must be tangent to AC at A. By PoP we find that $CA^2 = CD \cdot CB \Rightarrow AC = 2\sqrt{2}$.

Now our computations are straightforward:

$$\begin{aligned}\tan\angle BAD &= \frac{\sin\angle BAD}{\cos\angle BAD} = \frac{\frac{2\sin\angle ABD}{AD}}{\frac{AB^2+AD^2-BD^2}{2AB\cdot AD}}\\ &= \frac{4\sin\angle ABD\cdot AB}{AB^2+AD^2-4} = \frac{4AC\sin\angle ACB}{AB^2+AD^2-4}\\ &= \frac{4\sqrt{6}}{(4^2+(2\sqrt{2})^2-4\cdot 2\sqrt{2})+(2^2+(2\sqrt{2})^2-2\cdot 2\sqrt{2})-4} = \frac{4\sqrt{6}}{32-12\sqrt{2}}\\ &= \frac{\sqrt{3}}{4\sqrt{2}-3}\end{aligned}$$

3. What are the chances?

Problem 9. (**10 points**) Fifty people line up to enter a movie theater. Each has an assigned seat. However, the first person to enter has lost her movie ticket and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to enter gets to sit in his assigned seat?

Solution. When the last person enters, the only possible open seat is either his own or that of the first person. If you swapped the assigned seats of the first and last people, nothing in this problem would change. Therefore, the last person is just as likely to be left with the first person's seat as his own, so the probability is $\frac{1}{2}$.

Problem 10. (**5 points first part, 10 points second part**) Kelp Kelp works on the 17th floor of a 20 floor building. The only elevator moves continuously through floors 1, 2, . . . , 20, 19, . . . , 2, 1, 2, . . . , except that it stops on a floor on which the button has been pressed. Assume that time spent loading and unloading passengers is very small compared to the travelling time.
Kelp Kelp complains that when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation?
Now assume that the building has 2 elevators, which move independently. What is the probability that the first elevator to stop on the 17th floor is going down?

Solution. It will only go down if he pressed the button when it was going up from 17 to 20 or down from 20 to 17, which is $\frac{6}{40} = 0.15$. So therefore it seems like it's always going up. (Not always, Kelp Kelp is just a drama queen)

Answer for second part is **25.5%** (probably)
You integrate over the position of the first elevator and compute the probability of winning in each case.

Problem 11. (**5 points**) Math Gherman has an unfair coin which lands on heads with probability $p \in (0, 1)$. Devise a game with the coin for which the probability of either side winning is $\frac{1}{2}$.

Solution. Flip the coin twice. If the two results are the same, start over. Otherwise, H-T and T-H are equally likely.

Problem 12. (**10 points**) The numbers $0, 1, 2, ..., 7$ are arranged clockwise on a circle. The world's fastest turtle starts at 0 and at each step moves at random to one of its two neighbors. For each i, compute the probability p_i that i is the last of the numbers to be visited. For example, $p_0 = 0$ since 0 is visited first.

Solution. It's $\frac{1}{7}$. At the first time the turtle is adjacent to point i, winning is equivalent to the turtle visiting the other neighbor before it visits i. This formulation is equivalent for all points, and therefore they are equal by symmetry.

Problem 13. (**10 points**) Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

Solution. There are 2^6 possible colorings of the cube. Consider the color that appears with greater frequency. The property obviously holds true if 5 or 6 of the faces are colored the same, which for each color can happen in $6 + 1 = 7$ ways. If 4 of the faces are colored the same, there are 3 possible cubes (corresponding to the 3 possible ways to pick pairs of opposite faces for the other color). If 3 of the faces are colored the same, the property obviously cannot be satisfied. Thus, there are a total of $2(7 + 3) = 20$ ways for this to occur, and the desired probability is $\frac{20}{2^6} = \frac{5}{16}$.

4. Number stuff

Problem 14. (**5 points**) Let p be a prime. Show that p divides $\binom{p}{k}$ for all integers k between 1 and $p - 1$.

Solution. We know $\binom{p}{k}$ is an integer, call it n. Then we have $p! = nk!(p-k)!$. So p divides $nk!(p-k)!$, but it doesn't divide $k!(p-k)!$ since it's prime. Therefore, it divides n.

Problem 15. (**10 points**) Let $k = 2020^2 + 2^{2020}$. What is the units digit of $k^2 + 2^k$?

Solution. It's clearly even. So we must look mod 5. k is $0 + 1 = 1$ mod 5, and so k^2 is also 1 mod 5. Next, notice that 2^n is -1 mod 5 if n is 2 mod 4, and 1 mod 5 if $n \equiv 0$ mod 4. Since k is 0 mod 4, we have that 2^k is 1 mod 5. Therefore, overall we are 2 mod 5 and 0 mod 2, which gives us 2 mod 10, so the units digit is 2.

Problem 16. (**10 points**) The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

Solution. (2010 AMC 12A Problem 23) Let P be the result of dividing 90! by tens such that P is not divisible by 10. We want to consider $P \mod 100$. But because 100 is not prime, and because P is obviously divisible by 4 (if in doubt, look at the answer choices), we only need to consider P mod 25.

However, 25 is a very particular number. $1 * 2 * 3 * 4 \equiv -1 \mod 25$, and so is $6 * 7 * 8 * 9$. How can we group terms to take advantage of this fact?

There might be a problem when you cancel out the 10s from 90!. One method is to cancel out a factor of 2 from an existing number along with a factor of 5. But this might prove cumbersome, as the grouping method will not be as effective. Instead, take advantage of inverses in modular arithmetic. Just leave the negative powers of 2 in a "storage base," and take care of the other terms first. Then, use Fermat's Little Theorem to solve for the power of 2.

We will use the fact that for any integer n,

$$\begin{aligned}(5n+1)(5n+2)(5n+3)(5n+4) &= [(5n+4)(5n+1)][(5n+2)(5n+3)]\\ &= (25n^2+25n+4)(25n^2+25n+6) \equiv 4\cdot 6\\ &= 24 \pmod{25} \equiv -1 \pmod{25}.\end{aligned}$$

First, we find that the number of factors of 10 in 90! is equal to $\lfloor \frac{90}{5} \rfloor + \lfloor \frac{90}{25} \rfloor = 18+3 = 21$. Let $N = \frac{90!}{10^{21}}$. The n we want is therefore the last two digits of N, or $N \pmod{100}$. If instead we find $N \pmod{25}$, we know that $N \pmod{100}$, what we are looking for, could be $N \pmod{25}$, $N \pmod{25} + 25$, $N \pmod{25} + 50$, or $N \pmod{25} + 75$. Only one of these numbers will be a multiple of four, and whichever one that is will be the answer, because $N \pmod{100}$ has to be a multiple of 4.

If we divide N by 5^{21} by taking out all the factors of 5 in N, we can write N as $\frac{M}{2^{21}}$ where

$$M = 1\cdot 2\cdot 3\cdot 4\cdot 1\cdot 6\cdot 7\cdot 8\cdot 9\cdot 2\cdots 89\cdot 18,$$

where every multiple of 5 is replaced by the number with all its factors of 5 removed. Specifically, every number in the form $5n$ is replaced by n, and every number in the form $25n$ is replaced by n.

The number M can be grouped as follows:

$$\begin{aligned}M =& (1\cdot 2\cdot 3\cdot 4)(6\cdot 7\cdot 8\cdot 9)\cdots(86\cdot 87\cdot 88\cdot 89)\\ &\cdot (1\cdot 2\cdot 3\cdot 4)(6\cdot 7\cdot 8\cdot 9)\cdots(16\cdot 17\cdot 18)\\ &\cdot (1\cdot 2\cdot 3).\end{aligned}$$

Where the first line is composed of the numbers in 90! that aren't multiples of five, the second line is the multiples of five and not 25 after they have been divided by five, and the third line is multiples of 25 after they have been divided by 25.

Using the identity at the beginning of the solution, we can reduce M to

$$\begin{aligned}M &\equiv (-1)^{18}\cdot(-1)^3(16\cdot 17\cdot 18)\cdot(1\cdot 2\cdot 3)\\ &= 1\cdot -21\cdot 6\\ &= -1 \pmod{25} = 24 \pmod{25}.\end{aligned}$$

Using the fact that $2^{10} = 1024 \equiv -1 \pmod{25}$ (or simply the fact that $2^{21} = 2097152$ if you have your powers of 2 memorized), we can deduce that $2^{21} \equiv 2 \pmod{25}$. Therefore $N = \frac{M}{2^{21}} \equiv \frac{24}{2} \pmod{25} = 12 \pmod{25}$.

Finally, combining with the fact that $N \equiv 0 \pmod{4}$ yields $n = 12$.

Problem 17. (**10 points**) For an integer $n > 3$, denote by $n?$ the product of all prime numbers less than n. Find all solutions to $n? = 2n + 16$ and prove that there are no others. (Hint: You can use Bertrand's postulate, which says that there always exists a prime number between n and $2n$).

Solution. $n = 7$ works by inspection. Now we prove that no numbers greater than 7 will work, and you can try the smaller ones to see they fail. Suppose We have $n? = 2n + 16$. Since there is a prime between n and $2n$, that means $(2n)? \geq n * n? = n * (2n+16) = 2n^2 + 16n > 4n + 16 = 2(2n) + 16$. In fact, $(2n)? \geq n * n? > 2(4n) + 16$. Therefore, $(2n)? - (2(4n)+16) > 0$. Since $n?$ is increasing, this means that there are no solutions from $2n$ to $4n$. Repeating the argument with $2n$ instead of n, we get no solutions from $4n$ to $8n$, and so on and so on. Therefore, the only possible solutions would come between 7 and 14. But since 7 is prime, we already have that the next one will be too big, so there are no solutions there either.

Problem 18. (**10 points**) Prove that there are no integer solutions to $a^2 - b^2 = 2021 + 27k$ for any integer k.

> **Solution.** Reducing mod 9, we show that there are no solutions to $a^2 - b^2 \equiv 5 \bmod 9$. The squares mod 9 are $0, 1, 4$ and 7. There is no way to subtract two of those to get 5 mod 9.

5. Miscellaneous

Problem 19. (**10 points**) What is the minimum value of $f(x) = |x - 1| + |2x - 1| + |3x - 1| + \cdots + |119x - 1|$?

> **Solution.** (2010 AMC 12A Problem 22) If we graph each term separately, we will notice that all of the zeros occur at $\frac{1}{m}$, where m is any integer from 1 to 119, inclusive: $|mx - 1| = 0 \implies mx = 1 \implies x = 1/m$.
>
> The minimum value of $f(x)$ occurs where the absolute value of the sum of the slopes is at a minimum ≥ 0, since it is easy to see that the value will be increasing on either side. That means the minimum must happen at some $\frac{1}{m}$.
>
> The sum of the slopes at $x = \frac{1}{m}$ is
>
> $$\sum_{i=m+1}^{119} i - \sum_{i=1}^{m} i$$
> $$= \sum_{i=1}^{119} i - 2\sum_{i=1}^{m} i$$
> $$= -m^2 - m + 7140$$
>
> Now we want to minimize $-m^2 - m + 7140$. The zeros occur at -85 and 84, which means the slope is 0 where $m = 84, 85$.
>
> We can now verify that both $x = \frac{1}{84}$ and $x = \frac{1}{85}$ yield 49.

Problem 20. (**5 points**) Let a, b, c be real numbers. Show that at least one of the equations $x^2 + (a - b)x + (b - c) = 0, x^2 + (b - c)x + (c - a) = 0, x^2 + (c - a)x + (a - b) = 0$ has a real solution.

> **Solution.** A real quadratic has a real solution if it's discriminant is non-negative. Therefore, we'd like to show that at least one of the discriminants is non-negative. Adding all three discriminants together, we get $(a-b)^2 + (b-c)^2 + (c-a)^2$, which is non-negative. If the sum of three real numbers is non-negative, then at least one of the numbers must be non-negative.

Problem 21. (**5 points**) Yelmer Fuzz, Sir Wellington Snack, Tim Colitis, and Orin Shank need to cross a dark river at night. They only have one torch and the river is too spooky to cross without the torch. If more than two cross simultaneously, then the torch light won't suffice and they'll all get spooked to oblivion. Every person crosses the river at a different speed. Yelmer crosses in 1 minute, it takes Sir Snack 2 minutes, Tim Colitis takes 7 minutes, and as we all know, Orin Shank takes a full 10 minutes to cross. What is the shortest time needed for all four of them to cross the river ?

> **Solution.** 17 minutes. 1 and 2 cross, 2 comes back, 7 and 10 cross, 1 comes back, 1 and 2 cross.

Problem 22. (**10 points**) The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of $a + b$?

Solution. $\frac{y}{x}$ represents the slope of a line passing through the origin. It follows that since a line $y = mx$ intersects the ellipse at either $0, 1,$ or 2 points, the minimum and maximum are given when the line $y = mx$ is a tangent, with only one point of intersection. Substituting,

$$2x^2 + x(mx) + 3(mx)^2 - 11x - 20(mx) + 40 = 0$$

Rearranging by the degree of x,

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0$$

Since the line $y = mx$, we want the discriminant,

$$(20m + 11)^2 - 4 \cdot 40 \cdot (3m^2 + m + 2) = -80m^2 + 280m - 199$$

to be equal to 0. We want $a + b$, which is the sum of the roots of the above quadratic. By Vieta's formulas, that is $\frac{280}{80} = \frac{7}{2}$

Problem 23. (**10 points**) Show that there's a number consisting of only 0s and 1s with 2021 digits or fewer which is divisible by 2021.

Solution. Look at the residues of $1, 11, 111, ...$ mod 2021 and use pigeonhole. If two are the same, then subtract the smaller from the larger.

Problem 24. (**10 points**) For any positive integer n, show that there are n consecutive numbers, all of which are composite.

Solution. Prime number theorem $\#PNT\#FTW\#DensityArguments\#KillAnAntWithaHammer$ ... or Alex Wertheim says $(n+1)! + 2, (n+1)! + 3, ..., (n+1)! + (n+1)$

Problem 25. (**10 points**) Write 271 as a sum of positive real numbers so as to maximize their product.

Solution. We know for two numbers, the product is maximized when they are equal. We can extend this to n numbers. So if we break it into n numbers, each will be $\frac{271}{n}$, giving a product of $(\frac{271}{n})^n$. Setting the derivative with respect to n equal to 0, we get $n = 271/e$, which is closest to 100. Therefore, we split it into 100 copies of 2.71.

Problem 26. (**10 points**) The plane is divided into areas by some number of straight infinite lines. Show that these areas can be colored using only two colors, so that any two states that share a border line, have different colors.

Solution. For each line, label one side 0 and the other 1. Then each region will be the sum of the 0s and 1s corresponding to all the lines. Take this number mod 2 and that determines the colors. Crossing a line changes this sum by 1 mod 2 and therefore adjacent regions have different colors.