LET'S GET AFTER IT!

OLGA RADKO MATH CIRCLE
ADVANCED 2
JANUARY 10, 2021

1. Hats, Bats, Rats, Hats

Problem 1. (**10 points**) Three people are each given a red or a blue hat at random. Each one can see the other people's hats, but not their own. They are told to raise their hands if they see someone wearing a red hat, and a prize is offered to the first person to (correctly) guess their hat color.
All three raise their hands, and several minutes pass. Then somebody guesses "My hat is red", winning the prize.
How did they know, and what colors were the other hats?

The following are interactive hat problems. To play one of these problems, please read the problem carefully, agree on a strategy, and then gather a group, and head over to the main breakout room to play.

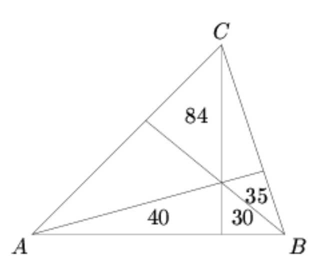
Problem 2. (**20 points**) Please gather a group of at least 6.
The instructor will assign each of you a hat color, red, green, or blue, at random. IRL, we might put a real hat on your head with your eyes closed, but today, the instructor will message everyone except you your hat color in the chat.
Once everybody is ready, then there are two rounds of guessing. In the first round, the instructor randomly chooses one of you, who guesses their hat color, which is allowed to be incorrect, but they are not told the answer. After two minutes of thinking independently, in the second round, everyone except that first representative must simultaneously guess their hat colors. This second time, you must all get it right.

Problem 3. (**20 points**) Please gather a group of 3 people.
Each of you will be assigned a red or blue hat, and as before, you will only be told everybody else's hat color.
There is only one round of guessing, you all must guess your hat color, or pass, at once. You win as a group if not everyone passes, and everyone who doesn't pass guesses their hat color correctly.
Because this game is faster, you can play up to 4 times, and you get 5 point for each win

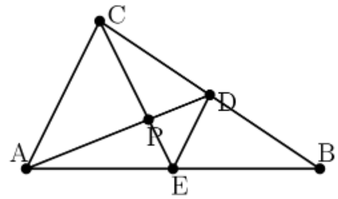
2. Gee, I'm a tree!

Problem 4. (**10 points**) Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

Problem 5. (**10 points**) Triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



Problem 6. (**10 points**) In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



Problem 7. (**10 points**) Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

Problem 8. (**10 points**) Triangle ABC has $\angle C = 60°$ and $BC = 4$. Point D is the midpoint of BC. What is the largest possible value of $\tan \angle BAD$?

3. What are the chances?

Problem 9. (**10 points**) Fifty people line up to enter a movie theater. Each has an assigned seat. However, the first person to enter has lost her movie ticket and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to enter gets to sit in his assigned seat?

Problem 10. (**5 points first part, 10 points second part**) Kelp Kelp works on the 17th floor of a 20 floor building. The only elevator moves continuously through floors 1, 2, . . . , 20, 19, . . . , 2, 1, 2, . . . , except that it stops on a floor on which the button has been pressed. Assume that time spent loading and unloading passengers is very small compared to the travelling time.
Kelp Kelp complains that when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation?
Now assume that the building has 2 elevators, which move independently. What is the probability that the first elevator to stop on the 17th floor is going down?

Problem 11. (**5 points**) Math Gherman has an unfair coin which lands on heads with probability $p \in (0, 1)$. Devise a game with the coin for which the probability of either side winning is $\frac{1}{2}$.

Problem 12. (**10 points**) The numbers $0, 1, 2, ..., 7$ are arranged clockwise on a circle. The world's fastest turtle starts at 0 and at each step moves at random to one of its two neighbors. For each i, compute the probability p_i that i is the last of the numbers to be visited. For example, $p_0 = 0$ since 0 is visited first.

Problem 13. (**10 points**) Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

4. Number stuff

Problem 14. (**5 points**) Let p be a prime. Show that p divides $\binom{p}{k}$ for all integers k between 1 and $p - 1$.

Problem 15. (**10 points**) Let $k = 2020^2 + 2^{2020}$. What is the units digit of $k^2 + 2^k$?

Problem 16. (**10 points**) The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

Problem 17. (**10 points**) For an integer $n > 3$, denote by $n?$ the product of all prime numbers less than n. Find all solutions to $n? = 2n + 16$ and prove that there are no others. (Hint: You can use Bertrand's postulate, which says that there always exists a prime number between n and $2n$).

Problem 18. (**10 points**) Prove that there are no integer solutions to $a^2 - b^2 = 2021 + 27k$ for any integer k.

5. Miscellaneous

Problem 19. (**10 points**) What is the minimum value of $f(x) = |x-1| + |2x-1| + |3x-1| + \cdots + |119x-1|$?

Problem 20. (**5 points**) Let a, b, c be real numbers. Show that at least one of the equations $x^2 + (a-b)x + (b-c) = 0, x^2 + (b-c)x + (c-a) = 0, x^2 + (c-a)x + (a-b) = 0$ has a real solution.

Problem 21. (**5 points**) Yelmer Fuzz, Sir Wellington Snack, Tim Colitis, and Orin Shank need to cross a dark river at night. They only have one torch and the river is too spooky to cross without the torch. If more than two cross simultaneously, then the torch light won't suffice and they'll all get spooked to oblivion. Every person crosses the river at a different speed. Yelmer crosses in 1 minute, it takes Sir Snack 2 minutes, Tim Colitis takes 7 minutes, and as we all know, Orin Shank takes a full 10 minutes to cross. What is the shortest time needed for all four of them to cross the river ?

Problem 22. (**10 points**) The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of $a + b$?

Problem 23. (**10 points**) Show that there's a number consisting of only 0s and 1s with 2021 digits or fewer which is divisible by 2021.

Problem 24. (**10 points**) For any positive integer n, show that there are n consecutive numbers, all of which are composite.

Problem 25. (**10 points**) Write 271 as a sum of positive real numbers so as to maximize their product.

Problem 26. (**10 points**) The plane is divided into areas by some number of straight infinite lines. Show that these areas can be colored using only two colors, so that any two states that share a border line, have different colors.