ORMC: BOARD PROBLEMS (TILINGS, COLORINGS, AND MORE)

OLYMPIAD GROUP 1, WEEK 8

Problem 1. (Textbook examples)

(a) We take a chessboard (8×8) and remove two opposite corners. Show that one cannot tile the remaining 62 squares with dominoes (or) completely and without overlaps.

(b) Consider a 2021×2021 with a corner square removed. Can one tile the remaining squares with linear trominos (or) completely and without overlaps?

Problem *2. Consider a $(4n + 2) \times (4n + 2)$ board.

(a) Show that the board cannot be tiled with T-tetrominoes (and its rotations) completely and without overlaps.

(b) Show that the board cannot be tiled with L-tetrominoes (and its rotations) completely and without overlaps.

(c) Show that the tilings in parts (a) and (b) are possible for a $4n \times 4n$ board.

Problem 3. The numbers from 1 to 25 are written in a 5×5 board, one in each cell.

(a) Show that there exist two neighboring cells (i.e. sharing an edge) whose entries have distance at least 3.

(b) Show that there exist two neighboring cells whose entries have distance at least 4.

Problem 4. A 2021×2021 board is filled in with numbers from 1 to 2021, such that:

- Every row and every column contains distinct numbers;
- The board is symmetric with respect to the main diagonal, i.e. the (m, n)-th entry coincides with the (n, m)-th entry.

Show that the numbers on the main diagonal are all distinct.

Problem 5. We are given a 2020×2020 board, and we wish to paint half of its tiles in black, so that there are no two adjacent black tiles (sharing an edge). In how many ways can this be done? *Hint: tiling can be useful!*

Problem *6. The tiles of a 4×4 board are all colored in white initially. With every move, you change the colors of all tiles in a row or in a column (from white to black and viceversa). Can you ever reach a board with exactly 1 white tile? How about 2, 3, 4 and 5 white tiles?

Homework

Problem 1. Show that a 2021×2022 board cannot be tiled with fish-shapes (and its rotations), completely and without overlaps. *Hint: the solution is very similar to that of 2 (a).*

Problem 2. On an 8×8 board, we wish to paint some of the tiles in red, such that no two red tiles lie on the same diagonal. What is the greatest number of red tiles that we can paint? (Note: in the language of chess, how many bishops can you put on the board such that no two attack each other?)