Egyptian Multiplication and Binary Numbers

UCLA Olga Radko Math Circle 11/29/2020

Kingdoms of ancient Egypt span the part of human history that begins around 3100 BC and ends at 30 BC when Egypt was conquered by the Roman Empire. We're now interested in how they did multiplication! Well, let's first recall that **multiplication** was introduced as **repeated addition**. Let us also introduce a new operation that represents repeated multiplication, ***raising to a power*.**

Warm-Up

The following reads as ***b to the power of x****.*

$$b^x = \underbrace{b \times b \times \ldots \times b}_{x \ times}$$

1. *Solve the following problems:*

   a. 2^5 = ________32_____________

   b. 10^3 = _________1000_____________

   c. 10^4 = ____________10000_________

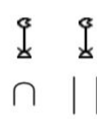
   d. $2^3 \times 2^2$ = 32

Egyptian Notation

The Egyptians were the teachers of the Greeks who in their turn laid the foundation for Western civilization in general and mathematics in particular. This is how the teachers of our teachers have written numbers.

1	10	100	1,000	10,000	100,000	1,000,000 or many
|	∩				 or 	

For example, they would have written 2012 as:



Problem 1: *Represent the following numbers in Egyptian Notation.*

a. 20 =

∩ ∩

b. 47 =

∩ ∩ ∩ ∩

| | | | | | |

c. *11,111 =*

Egyptian Multiplication

Our numeral system is **place-value base ten**. This means that we express numbers as sums of powers of ten, using ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as coefficients (the multiples in front of the powers).

For example, we can write the number 3641 as follows:

$$3641 = (\mathbf{3} \times 1000) + (\mathbf{6} \times 100) + (\mathbf{4} \times 10) + (\mathbf{1} \times 1)$$

Rewrite the previous expression in power notation:

$$3641 = (\mathbf{3} \times 10^{3}) + (\mathbf{6} \times 10^{2}) + (\mathbf{4} \times 10^{1}) + (\mathbf{1} \times 10^{0})$$

The **most efficient way** to multiply big numbers known today, ***long multiplication***, was unavailable to the Egyptians. Thousands of years earlier, ancient Egyptians had nothing like that available. So how did they multiply numbers? By ***doubling*** numbers.

<u>Example 1:</u>

Suppose they want to multiply two numbers (**23 by 51).** Here is what they did to complete the multiplication:

1. **Below the first number** (in this case 23), they would **start with 1** and keep doubling the number, until the results **exceed the first number**, 23.

	23	51
	– –	– –
	1	
23 x 51 :	2	
	4	
	8	
	16	
	32	

2. In the **second column**, they would write the **multiples of the second number** (in this case 51), **doubling it** in the next line. This produces 51,102, 204, 408, and 816.
   a. 32 is the first number in the left column greater than 23, so that's why we stop.

	23	51	*Rows Used*
	– –	– –	– –
	1	51	✔
23 x 51 :	2	102	✔
	4	204	✔
	8	408	
	16	816	✔
	32		

   b. *What do you notice about the numbers in the first column? (Remember, we used to call them basic binary numbers, in the past)*

   They are powers of two!

3. After that, they would **represent** the **first number** (23) as the **sum of numbers in the first column,** only using each number **once**:

$$23 = 16 + 4 + 2 + 1$$

4. After that, they would **mark** those **rows** where the **number** in the **first column is used**. (In our example it would be **the first**, **the second**, **the third**, and **the fifth** rows.)

5. Finally, all that is left to do is **add the marked numbers** in the **second column**:

$$\begin{array}{r} 51 \\ 102 \\ 204 \\ +\ 816 \\ \hline \end{array}$$

Therefore, **23 x 51** = 1173

Let's check with a calculator to see if this is correct!

Problem 2:

Use Egyptian Multiplication to multiply 13 and 22:

	13	22	*Rows Used*
	– –	– –	– –
	1	22	✔
13 x 22 :	2	44	
	4	88	✔
	8	176	✔
	16		

a. *Write the appropriate powers of two below the first column.*
b. *Double each row going down in the second column.*
c. *Represent 13 as a sum of powers of 2:*

13 = 8 + 4 + 1

d. *Finish the Egyptian Multiplication to find 13 times 22.*

= 286

e. *When representing a number as the sums of powers of 2, did you start with the highest or lowest power of 2? Why? (Try both to see which is easier).*

Start with the highest power of 2

f. *Check with a calculator. Did we obtain the correct answer? Yes!*

<u>Problem 3:</u>

a. *Solve the following problems using Egyptian Multiplication.*
   i. *13 x 41*

   ii. *41 x 13 =533*

b. *Given two numbers, which one (**smaller**, or **larger**) will you **use** as the **first number** in Egyptian Multiplication? Why?*

Smaller since we would end up using less powers of 2 leading to potentially less rows being used and thus less work.

c. *Check with a calculator. Did we obtain the correct answer? Yes!*

Note: *In Egyptian Multiplication, all it took to multiply two numbers were **doubling**, that is adding a number to itself.*

Egyptian and Decimal Place Value Bases

It seems like we have a miracle here! This technique, which **utilizes doubling**, produces the **correct answer** in our **multiplication problems**, but ***how?***

Let's continue by looking at the example problem. Recall that the first column contains powers of 2.

	23	51	*Rows Used*	
	— —	— —	— —	— —
	1	51	✔	51 = 1 x 51
23 x 51 :	**2**	102	✔	102 = 2 x 51
	4	204	✔	204 = 4 x 51
	8	408		408 = 8 x 51
	16	816	✔	816 = 16 x 51
	32			

Problem 4:

*Explain how each number in the **second column** is **obtained** from the number in the **first column**.*

a. *How do we get **102** from **2 and 51**? 51 x 2*

b. *How do we get **204** from **4 and 51**? 51 x 4*

c. *Continue and fill in the **4th column** in the **example above** (the first one has been done).*

Problem 5:

In Example 1, we completed the following sum: 51 + 102 + 204 + 816.

a. ***Rewrite** each term in the **sum** as a **product** of the **number** in the **first column** (a power of 2) and 51.*

i. 51 = 1 x 51

ii. 102 = 2 x 51

iii. 204 = 4 x 51

iv. 816 = 16 x 51

b. Now let's rewrite the sum we computed in the first Example:

1173 = 51 + 102 + 204 + 816 = (51 x 1) + (51 x 2) + (51 x 4) + (51 x 16)

c. Rewrite using Power Notation:

$(51 \times 2^0) + (51 \times 2^1) + (51 \times 2^2) + (51 \times 2^4)$

*d. Our numeral system is **place-value base ten** meaning that we express numbers as **sums of powers of ten**.*

i. What is the place-value base for Egyptian Multiplication?

Place-Value base 2.

ii. What does this mean?

We express numbers as sums of powers of 2.

e. What do you notice in the following expression?

i. Can you simplify it by factoring anything out?

(51 x 1) + (51 x 2) + (51 x 4) + (51 x 16)

ii. If you factor out the 51, what would that yield?

51 x (1 + 2 + 4 + 16) = 51 x (23)

f. What does this tell us about the procedure and why it produces the correct product?

We introduced multiplication as repeated addition. This procedure is doing the same thing, but adding multiples of 51 generated by powers of 2.

Egyptian Multiplication and Binary Numbers

In the past, we've explored another system that utilizes the **powers of 2**: Binary Numbers. Recall that in Egyptian Multiplication, we represent (positive integral) **numbers** as **sums of powers of two**. When we do this, we only get **two coefficients**, **0** and **1**. **Zero** means that the corresponding **power of two is missing** from the representation, **one** - it is **present**.

Example 2: *Represent 57 as the sum of powers of two.*

a. *To do this, we can utilize one of the steps from Egyptian Multiplication: writing down the powers of 2 until we exceed the given number.*

57

– –

1

2

4

8

16

32

64

b. *Write 57 as the sum of powers of 2:* ***57 = 32 + 16 + 8 + 1***

c. *In power notation:* **57 =** $2^5 + 2^4 + 2^3 + 2^0$

The following chart organizes the representation of the number 57:

n	0	1	2	3	4	5
2^n	1	2	4	8	16	32
coefficient	1	0	0	1	1	1

So, we can rewrite this as a *binary representation*:

$$57_{10} = 111001_2$$

Note: *The sub-index means that the base of the corresponding place-value system, 10 on the left-hand side, two - on the right.*

Problem 6:

a. *Find the binary representations of the following numbers:*

i. 5_{10} = ***101_2***

ii. 18_{10} = ***10010_2***

iii. 63_{10} = ***111111_2***

b. *Find the decimal representations of the following numbers.*

i. 110_2 = ***6_{10}***

____

ii. 1000000_2 = ***64_{10}***

____

iii. 10110111_2 = ***183_{10}***

____

Problem 7:

*As we were exploring Egyptian Multiplication in **problem 5**, we saw that we can represent **23 x 51** as the following in power notation:*

$$(51 \times 2^4) + (51 \times 2^2) + (51 \times 2^1) + (51 \times 2^0)$$

a. *By factoring out the 51, what do you get? (Leave it in power notation form)*

$$51 \times (2^4 + 2^2 + 2^1 + 2^0)$$

b. Since 51 is in decimal notation, we can rewrite it as 51_{10}. Can you rewrite the sum of the powers of two in binary notation?

$$51_{10} \times (10111_2)$$

c. Can you rewrite 51_{10} in binary notation?

$$(110011_2) \times (10111_2)$$

d. Therefore, Egyptian Multiplication is in fact Binary Multiplication!

<u>Problem 8:</u> *Solve the following problems using Egyptian Multiplication.*

a. 25 x 31 = 775

b. 38 x 45 = 1710

c. 112 x 85 = 9520

d. 52 x 235 = 12220