Exponential Functions Part I

UCLA Olga Radko Math Circle Beginners 2

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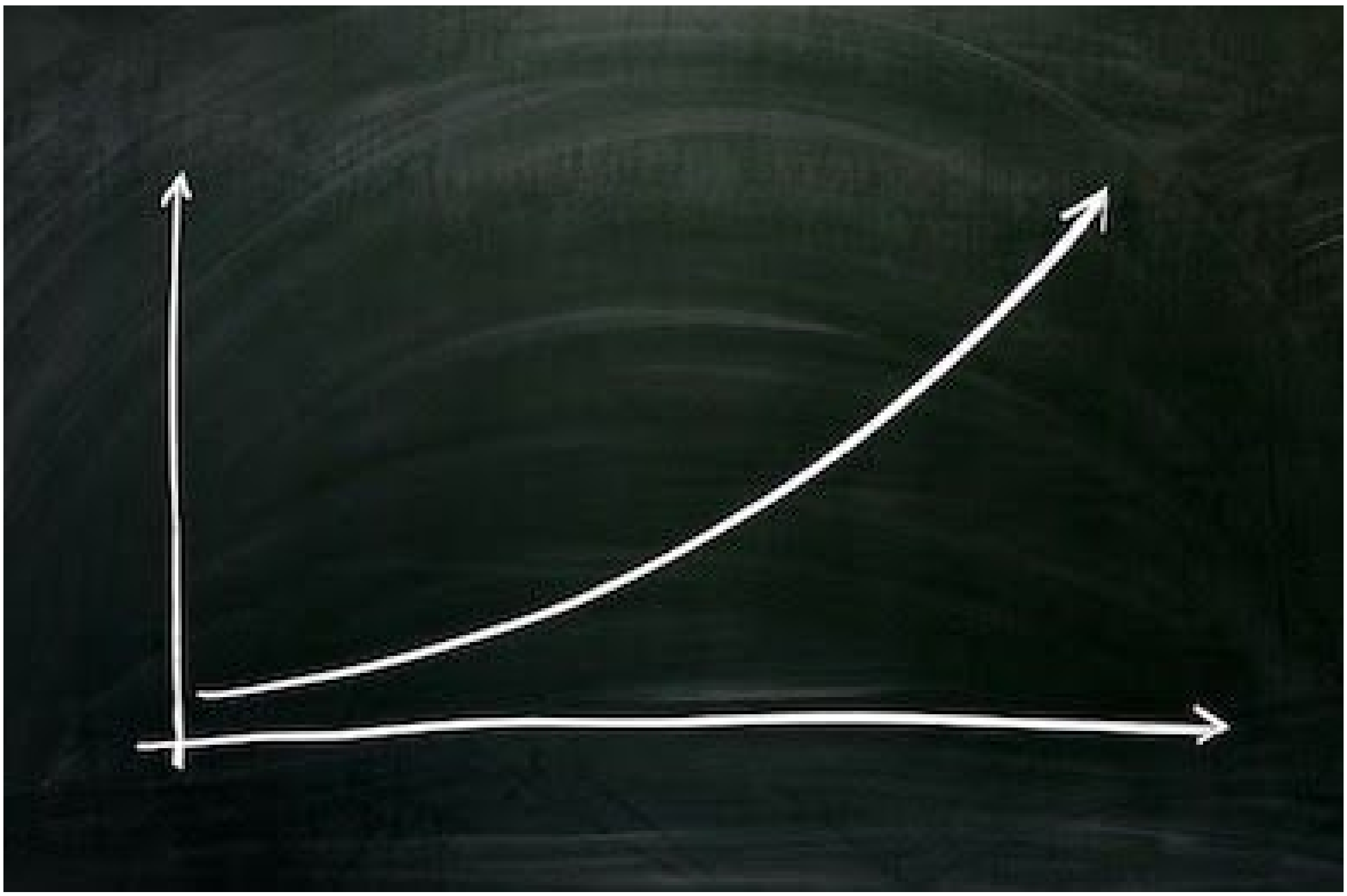

We previously studied Egyptian Math, where we discussed powers of 2 and 10. Today, we will dive deeper into the world of powers and exponents.

Warm-Up

Alice, Bob, Cole, and Donna were picking mushrooms. Alice picked more mushrooms than other children. There was a child who picked fewer mushrooms than Donna.

a. *Is it true that the girls picked more mushrooms than the boys? Circle the correct answer.*

True ***False***

b. *Explain your choice.*

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Exponential Functions

Problem 1: Most of the time, exponents are just a shorthand for multiplication. See if you can write out the following examples using *multiplication* instead of *exponents*.

a. What is the value of 2^4?

    i. 2^4 = _______________ =

b. What about the value of 3^5?

    i. 3^5 = _______________ =

c. What about 4^4?

    i. 4^4 = _______________ =

d. Let's try a challenging one. 5^{2021}?

    i. 5^{2021} =

    ii. We know that the value of 5^{2021} is going to be a ***really*** big number. So, let's try to use a calculator to find its value. What happens when we use a calculator to find its value?

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iii. How can exponents help us with this problem? (Why might exponents be useful in writing down really big numbers?)

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The Base

Problem 2: Now, let's try to generalize this. What might $\boldsymbol{b}^{2021}$ mean? (***b*** is a **natural number**).

a. So far, we have been considering **natural numbers**. What is a **natural number**?

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b. Is 0 a **natural number**? Explain.

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i. If we defined ***b*** to be a **natural number**, based on your previous answer, what can you say about ***b***?

____

c. What would happen if we let ***b*** = 1?

i. $\boldsymbol{b}^2 = (\ \ )^2 =$ ________________

ii. $\boldsymbol{b}^6 = (\ \ )^6 =$ ________________

iii. $\boldsymbol{b}^{2020} = (\ \ )^{2020} =$ ________________

iv. What happens when we let ***b*** = 1?

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d. Now, let's consider negative 1 (***b*** = -1).

   i. $\boldsymbol{b^2}$ = ( )^2 = ________________

   ii. $\boldsymbol{b^3}$ = ( )^3 = ________________

   iii. $\boldsymbol{b^4}$ = ( )^4 = ________________

   iv. $\boldsymbol{b^5}$ = ( )^5 = ________________

   v. $\boldsymbol{b^{12}}$ = ( )^{12} = ________________

   vi. $\boldsymbol{b^{13}}$ = ( )^{13} = ________________

   vii. Do you see a pattern? If so, what do you notice?

   ____

   viii. What happens when **b** is negative? *(Try doing the same problems but when **b = 1**)*

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e. The ***b*** we have explored so far (the *factor* being multiplied by itself) is called a **base.** Based on your answers from a-d, what kind of numbers can we use as the **base** for an exponential function?

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f. Based on everything we have learned so far, what might $\boldsymbol{b^{2021}}$ mean?

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Red Chilli Pepper Problem

A few sparrows decided to land on a line of trees along a street. Trying to land one bird on a tree, they found out that there were not enough trees for four sparrows. They landed two birds on one tree instead and one tree remained unoccupied. How many sparrows and how many trees were there?

The Exponent/Power

Problem 3: The **exponent/power** is the small number on the top right of our exponential functions. Based on the problems we have done so far, what does the **power** tell us?

As we continue with our exploration of powers, you'll see that the exponential function has two very important properties. Let's take a look at the first one through some examples.

Problem 4:

a. $4^2 \times 4^2 =$ ____ =

b. $2^3 \times 2^2 =$ ____ =

c. $3^4 \times 3^5 =$ ____ =

d. $6^{10} \times 6^{20} =$ ____ =

e. $5^{2021} \times 5^{100} =$ ____ =

f. What do you notice about what we have done so far?

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Problem 5: Let's see how we can generalize what we found. Suppose we have two exponents ***n*** and ***m***. Again, we are considering natural numbers for our exponents.

a. $2^n \times 2^m$ = ______________________ =

b. We've generalized our **base** to be represented as **b.** So what does $b^n \times b^m$ *equal*?

$b^n \times b^m$ = ______________________ = ____________________

Thus, we have found one of the properties of an exponential function. Let's take a look at another property

Problem 6: *Try these problems.*

a. $(2^2)^2$ = ____ =

b. $(3^4)^2$ = ____ =

c. $(4^3)^4$ = ____ =

d. $(5^{2021})^2$ = ____ =

e. What do you notice about what we have done so far?

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Problem 7: Let's see how we can generalize what we found. Suppose we have two exponents ***n*** and ***m***. Again, we are considering natural numbers for our exponents.

c. $(2^n)^m$ = ______________________ =

d. We've generalized our **base** to be represented as **b.** So what does the following *equal*?

$(b^n)^m$ = ______________________ = ___________________

Thus, we have found another property of an exponential function. These two properties are very important because as we continue to learn more about exponential functions, these two properties **must** hold true.

Comparing Numbers

Problem 8: Using everything we have learned so far to circle the larger value.

a. *What if they have the **same base**?*
   i. 2^5 vs 2^8
   ii. 3^7 vs 3^{10}
   iii. 4^{15} vs 4^8
   iv. In order for 2^m to be larger than 2^n, what must be true of **m** and **n**?

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b. *What if they have a **different base**? Is there something we can do to make them have the **same base**?*

   i. 8^2 vs 2^4

ii. 4^3 vs 16^9

iii. 125^2 vs 5^3

iv. 81^4 vs 3^5

c. *What if they have the **same exponent?***

i. 3^5 vs 2^5

ii. 3^{2021} vs 5^{2021}

iii. $3^5 \times 5^5$ vs $2^5 \times 8^5$

iv. $9^6 \times 2^6$ vs $3^6 \times 7^6$

v. Suppose we have two different bases, **b** and **h** with the properties we have defined so far. In order for $\mathbf{b^9}$ to be larger than $\mathbf{h^9}$**,** what must be true of **b** and **h**?

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d. *(Challenge) What if they have **different exponents**? Is there something we can do to make them have the **same exponent**?*

i. 8^3 vs 5^9

ii. 16^2 vs 3^8

Something to think about: So far, we've looked at exponents that are natural numbers (1, 2, …). But what would happen if it was 0? or a negative number? or even a fraction?

Challenge Problems

1. (Today is Sunday). I have a cold. I came to school anyway and caused two friends to get sick. The next day, my two friends came to school and each infected two other friends. **If this pattern continues, how many kids will get sick on Friday?** *Make sure to write your answer in exponential form.*

   ____

   a. What about the next Friday?

      ____

   b. The one after that?

      ____

2. What is 2^{2^3} ?

3. Cassandra and Paul are playing a game, where Paul tries to figure out which of the following would be the biggest. Cassandra gives Paul one hint: *"**x** is some number <u>greater than zero</u> but <u>less than one.</u>"* Which one should Paul pick?

x^2	x^3
x^4	$-x$

4. If an asteroid traveling at a rate of $2 * 10^3$ miles per day is currently $8 * 10^{15}$ miles away from Earth, how many days would it take to reach earth? (distance = rate * time)

5. The speed of light is $3 * 10^8$ meters per second. If a single photon of light travels for 500 minutes, how far does it travel? (There are 60 seconds in a minute and distance traveled = speed * time traveled)