School year 2020-2021 Week 1 Winter Quarter: Introduction to Complex Numbers

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A Walk Through Number Systems and "Justification" of Complex numbers

The goal of this worksheet is to introduce complex numbers (also referred to as imaginary numbers) and to give you the tools to complete basic manipulations of them. To achieve this goal we will start by examining other number systems and see how they build on each other.

Natural Numbers

We now start with the Natural numbers, or counting numbers. These are the most basic numbers and they are used to count, or enumerate, any discrete amount of positive things. As a set we can see the Natural Numbers as

$$\{1, 2, 3, 4, \ldots\}$$

or as

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

where the fancy N is used to denote the set and used to refer to it.

Exercise 1

Give an example of a process in real life where the natural numbers can be used to count it. Bonus points if you can think of a *theoretically* infinite process that uses the natural numbers to count it.

Exercise 2

Consider the inequality $4 < x < 7$.
a) Explain this inequality using the set model.
b) Explain this inequality using the number line

Integers

Now, we move on to the integers or in symbols:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4\}$$

Now, this set is intuitively clear, but it represents a radical shift in thinking. The Natural numbers gave only a representation for "positive" objects, but the integers allow for the taking away of things. I hope it is clear that the Natural Numbers can be found *inside* the integers.

Exercise 3

Now, try to create an example or two or more to make physical sense of the ideas that a negative times a negative is positive, a positive times a positive is positive, and a negative times a positive is negative.

These mathematical facts can be said to be defining properties of $\mathbb{Z}$ meaning they are chosen by mathematicians, but they have good reason to be chosen that way, namely these rules are those we find consistent with our physical reality.

Rational Numbers

Now, we move on to the next revolution in thought on numbers and in many cases is the limit of what we need for calculations. The rationals are all the fractions, or ratios, of integers. Seeing as this is a bit more difficult to write down in a list we instead write it down as

$$\mathbb{Q} := \left\{ x : x = \frac{a}{b} \text{ where a and b are integers} \right\}$$

Also, one can "see" the Integers inside the Rational numbers as all the fractions with denominator 1 and thus the Natural numbers are inside the Rationals as well.

Exercise 4

Given:

$$8q_1 + 2 = 5$$

$$5q_2 + 7 = 1$$

$$7q_3 + 4 = 9$$

find q_1, q_2, q_3

Real Numbers

And now we talk about the Real Numbers, somehow more real than the others. These numbers are impossible to define as the numbers above, but the Real Numbers are numbers you have dealt with or will deal with many times. A very simple example of a real number that is not any of the other types of numbers previously mentioned is $\sqrt{2}$. I will prove this fact at the end of the class. Lastly, take it on trust that in fact all 3 of the previous types of numbers are inside the Real Numbers.

Exercise 5

Given a right triangle with side lengths a, b, c where c is the longest side we have the Pythagorean theorem which states that

$$a^2 + b^2 = c^2$$

Now, consider a square with side lengths 1, show that its diagonal has length $\sqrt{2}$

And finally, the Complex Numbers

Now, there is a great many things to think about with the previous types of numbers and entire fields of mathematics are dedicated to each (e.g. number theory), but we are going to take a jump to the Complex Numbers, a set of numbers that can be thought of as just one more level higher than real numbers. Each previous type of number can be seen as filling in holes of the previous type (integers providing negatives, rationals giving us the ability to break numbers down, and real numbers giving us an understanding of numbers that cannot be seen as fraction) and the Complex Numbers are no different.

The core question of Complex Numbers is what the heck is

$$\sqrt{-1}$$

or in other words, what number times itself gives -1. Try as you might none of the types of numbers before gives you an answer.

So, mathematicians were clever and decided to just say that

$$\sqrt{-1} = i$$

OR

$$i^2 = -1$$

Now, this addition of i to our real numbers is that added one level of complexity and we can say that the complex numbers are the set

$$\mathbb{C} := \{a + bi : \text{ a and b are real numbers}\}$$

All this means for you is that a complex number can be seen as $a + bi$ where in some sense the real number with an i attached cannot combine with a real number without an i and are thus left apart by a plus sign.

Operations with Complex Numbers and Lots of Practice

Now, lets begin messing with Complex Numbers!

Simplifying Square Roots and Powers of i

For this section keep in mind the following identity

$$\sqrt{-1} = i$$

Exercise 6

Lets start with a simple proof that

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

To begin give a name to $\sqrt{a \cdot b}$ and $\sqrt{a} \cdot \sqrt{b}$ lets say d and c respectively.

Now, ask yourself what d^2 and c^2 are. Can you see how to finish the proof from here?

Now the point of the above proof is to let us simplify complicated square roots.

For example: Consider

$$\sqrt{24}$$

There is clearly no whole number answer and one might leave this as is, but using the above proof we realize that

$$(1)\ 24 = 4 \cdot 6 \text{ and thus we have } (2)\ \sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4} \cdot \sqrt{6} = 2 \cdot \sqrt{6}$$

Exercise 7

Now you try!

Simply the following:

(1): $\sqrt{32}$

(2): $\sqrt{200}$

(3): $\sqrt{324}$

(4): $\sqrt{68}$

(5): $\sqrt{8}$

Exercise 8

Now, recalling that $-4 = -1 \cdot 4$ try to simplify the following: (1): $\sqrt{-8}$

(2): $\sqrt{-36}$

(3): $\sqrt{-124}$

(4): $\sqrt{123464}$

(Hint: You will need the symbol i that we talked about before)

Exercise 9

Recall the fact that

$$\sqrt{-1} = i$$

and that

$$i^2 = -1$$

Try to solve the following: (1): i^3

(2): i^4

(3): $i^(73)$

After the above problems you should have a stronger understanding of what i "does" and how to use it to make sense of the square roots of any negative number.

But, lets recall that any imaginary number is a number of the form $a + bi$ and what use would that thing be if we could not add, multiply, and divide two of them.

Exercise 10

First, make up two imaginary numbers, lets say $3 + 4i$ and $7 + 100i$ and write down what you think is the "correct" way to add them together?

Exercise 11

For the above did you get $10 + 104i$? If not that is okay, the way we define the addition of two complex numbers is by the standard addition of their real and imaginary parts. Lets give this a try by solving the following

(1) $(104.4 + 1.9i) + (13 + 12 \cdot \frac{1}{2}i)$

(2) $(102.1 + 1.7i) + (1.6 + 3i)$

(3) let x and y be real numbers in the following expression and see if you can solve for them. $(x + 4i) + (3x + yi) = 24 + 11i$

Exercise 12

Now, take the two imaginary numbers $3 + 4i$ and $7 + 100i$ again and think of what it might mean to *multiply* them together. I promise it is likely not what you initially think it is.

Exercise 13

Did you get $-379 + 328i$?

In this instance we use something called the FOIL rule or a special case of the distributive law which says that $a(c + b) = ac + ba$

So what does that mean for us? Well take complex numbers $3 + 4i$ and $7 + 100i$ and multiply them to have $(3 + 4i) \cdot (7 + 100i)$ which by the distrubitve law $= 3 \cdot (7 + 100i) + 4i \cdot (7 + 100i) = 3 \cdot 7 + 3 \cdot 100i + 7 \cdot 4i + 4i \cdot 100i = 21 + 300i + 28i + 400i^2 = 21 - 400 + 328i = -379 + 328i$.

Exercise 14

Now, try the following:

(1) $(3 + 2i) \cdot (4 + 3i)$

(2) $(11 + 3i) \cdot (8 + 2i)$

(3) $(14 + 2i) \cdot (3 + 2i)$

(4) $4 \cdot 13i$

Complex Conjugate and Division

In this section we will introduce two operations: conjugation and division. These are rather simple operations but are incredibly useful for the theory of complex analysis.

For complex number $a + bi$ the conjugate of it is simply $a - bi$ with the following notation $\overline{a + bi} = a - bi$

Exercise 15

Go ahead and try the following:

(1): $(\overline{3 + 4i}) \cdot (3 + 4i))$

(2): $(\overline{1 + 2i}) \cdot (3 + 2i))$

(3): $(\overline{a + bi}) \cdot (a + bi))$

What you should notice is that a complex number multiplied to its conjugate gives a real number, this will be very helpful for division.

Exercise 16

First, what do you think would be the answer to $\frac{4+3i}{3}$? Then, what do you think would be the answer to $\frac{4+3i}{3+i}$

Exercise 17

The first one from above is a bit easier than the second. The answer to that division would be $\frac{4}{3} + i$.

As for the second we have to try something a bit different. We notice that dividing a complex number by a real one is understandable, and we know a way of turning a complex number into a real one, namely by multiplying by its conjugate. So, lets walk through that.

I leave it to you to notice that $(\overline{3 + i}) \cdot (3 + i) = 10$ and thus if we multiply $\frac{4+3i}{3+i}$ by $\frac{3-i}{3-i}$ we get $\frac{(4+3i)\cdot(3-i)}{10}$ which gives us $\frac{15+5i}{10}$ which we can solve.

Now try the following:

(1): $\frac{11+i}{3+i}$

(2): $\frac{3+6i}{3+2i}$

(3): $\frac{9+2i}{1+i}$