Winter Break Homework

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1 Problems - Geometry

1. Let I be incenter of triangle ABC. Show that circumcenters of triangles IAB, IBC, and ICA lie on a circle whose center is the circumcenter of triangle ABC.
2. What can you deduce in the degenerate case of Brianchon when the hexagon is a pentagon? What about the case when it degenerates to a quadrilateral?
3. Solve the following problem using Brianchon:

   (USAJMO 2011/5) Points A, B, C, D, E lie on circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} || \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.
4. The incircle of triangle ABC is tangent to BC, CA, AB at D, E, F respectively. Let M and N be the midpoints of BC and AC, respectively. Ray BI meets line EF at K. Show that $BK \perp CK$. Then show that K lies on line MN.
5. (USAMO 2010/1) Let $AXYZB$ a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.
6. (IMO 2000/1) Two circles G_1, G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.
7. In a cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of triangles ABC and BCD. Show that $AXYD$ is a parallelogram.
8. Let ABC be a triangle with circumcircle Γ. Let X be the intersection of the line BC with the tangent to Γ at A. Define Y and Z similarly. Show that X, Y, Z are collinear.
9. Let $ABCD$ be a cyclic quadrilateral and apply Pascal's theorem to $AABCCD$ and $ABBCDD$. What do we discover?
10. (Singapore TST) Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90$. Let P be any point on the tangent to ω at A other than A, and suppose ray PB intersects ω again at D. Point E lies on line CD so that $AE || BC$. Prove that P, O, E are collinear.
11. (Iran TST 2011/1) In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M midpoint of BC, let E and F be the feet of the altitudes from B and C respectively. Let K and L be the midpoints of ME and MF, respectively, and let T be on the line KL so that $TA || BC$. Prove that $TA = TM$. (Hint: Radius of circle zero.)