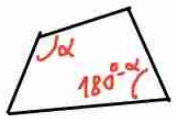
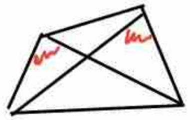
Angle Chasing

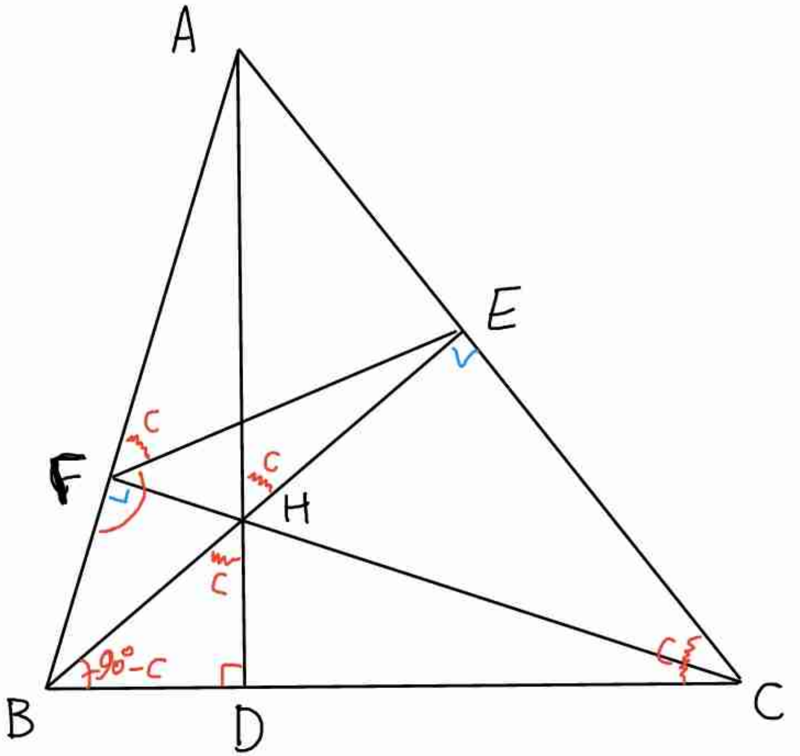
Recap: cyclic quadr.





Appl.: Orthocenter is well-def.: ($\triangle ABC$ acute)

↳ inters. of heights



Take 2 heights: BE, CF

Let $H := CF \cap BE$

$D := AH \cap BC$

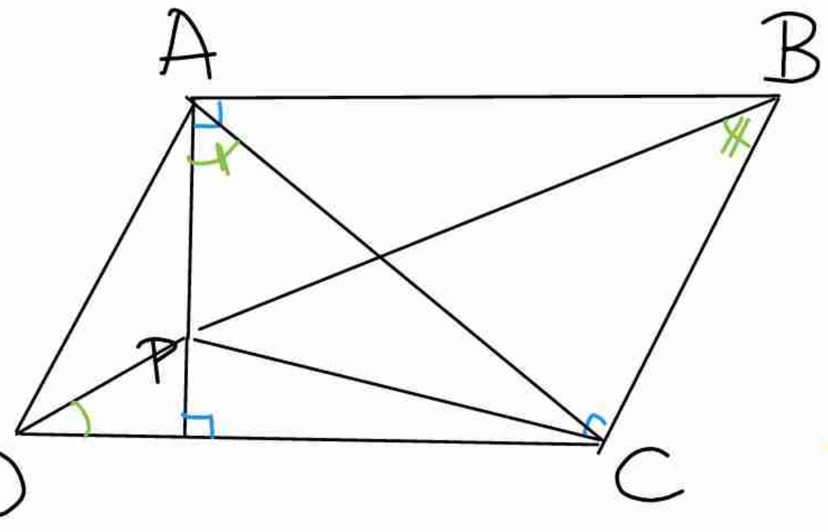
Goal: $\boxed{AD \perp BC}$

AEHF cyclic

BFEC cyclic ($\widehat{BFC} = \widehat{BEC} = 90^\circ$)

$\widehat{AFE} = 180^\circ - \widehat{BFE} = \widehat{BCE} = \hat{C}$

$\widehat{EBC} = 90^\circ - C$ (from $\triangle BEC$) $\Rightarrow \widehat{BDH} = 90^\circ$



Prove: $\widehat{PDC} = \widehat{PBC}$

$\begin{cases} PA \perp AB, \ PC \perp BC \\ AB \parallel DC \quad BC \parallel AD \end{cases}$

$\Downarrow$ $PA \perp DC$ $\qquad$ $\Downarrow$ $PC \perp AD$

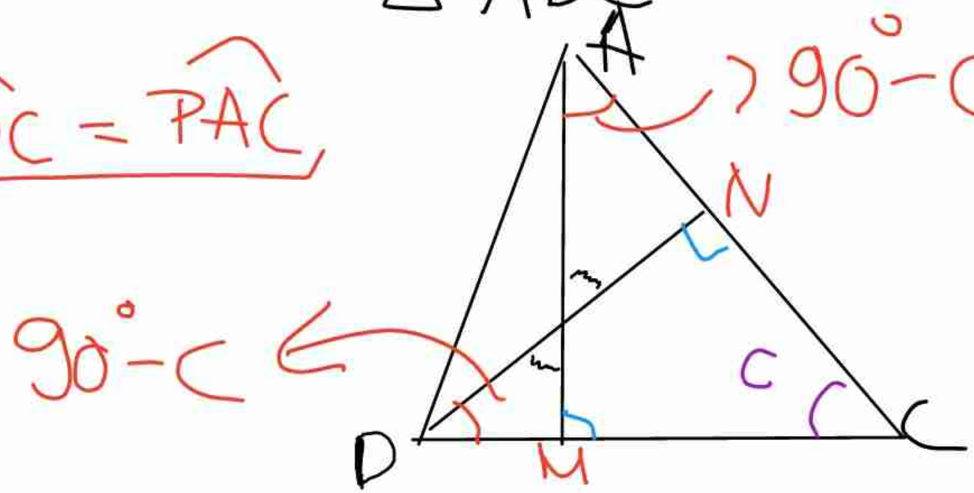
Since $\widehat{PAB} + \widehat{PCB} = 90^\circ + 90^\circ = 180^\circ$

$\Rightarrow$ PABC cyclic quad

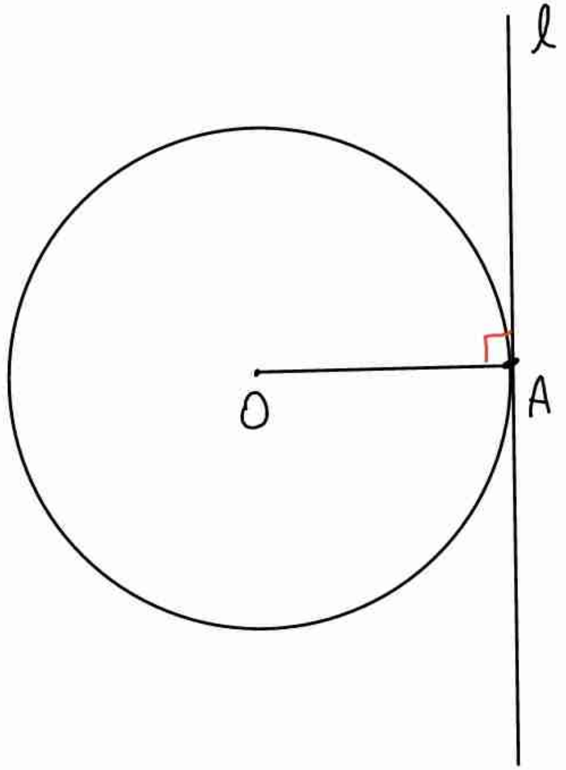
$\Rightarrow \widehat{PBC} = \widehat{PAC}$

$\Rightarrow$ P is orthocenter in $\triangle ADC$

Want: $\underline{\widehat{PDC} = \widehat{PAC}}$



Tangents to circles



A tangent to the circle is a line l which only touches the circle at one point.

(tangent at A $\longrightarrow$ that point is A)

Claim: A is on circle of center O, l line through A, then:

l is tangent to the circle $\iff$ $OA \perp l$

Proof of Claim:

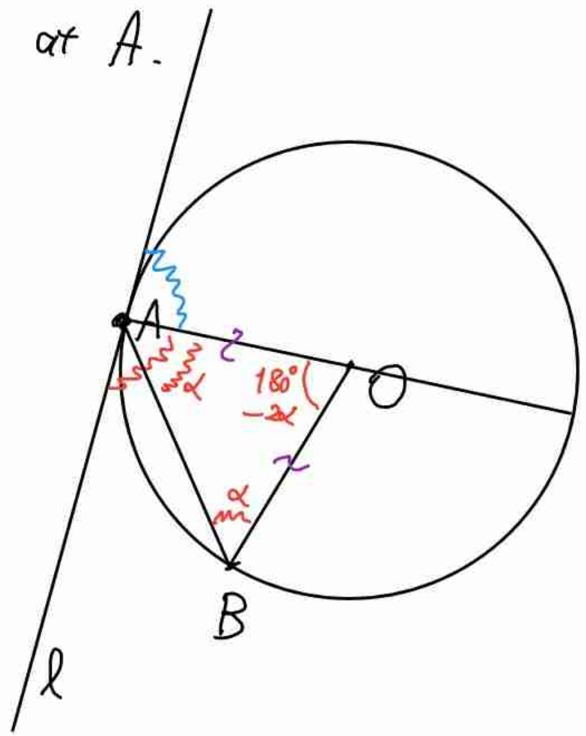
($\Rightarrow$) Suppose ℓ is tangent to the circle at A.

Suppose for the sake of contr. $OA \not\perp \ell$

$\Rightarrow$ Red angle or blue angle $< 90^\circ$,

Say red angle $=: \underline{\alpha < 90^\circ}$

If $\alpha = 90^\circ$, $A\hat{O}B = 0^\circ$ $\Rightarrow A = B$



Let B on circle with

$A\hat{O}B = 180^\circ - 2\alpha$ (on the same side of OA as the red angle)

> 0

But then, $O\hat{A}B = O\hat{B}A$

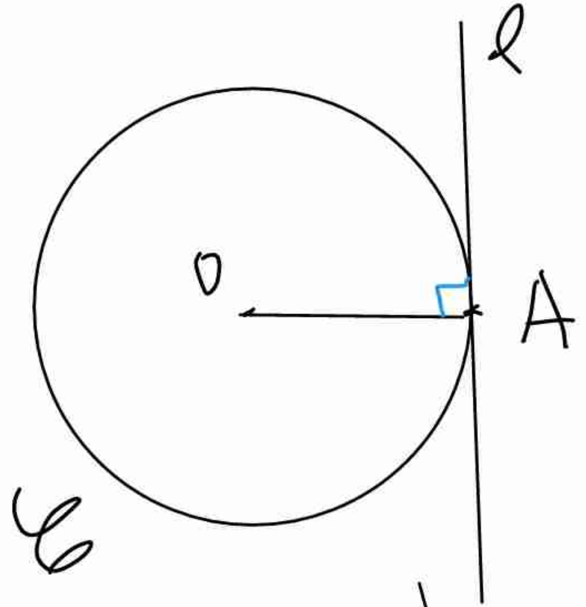
and $O\hat{A}B + O\hat{B}A + \underbrace{180^\circ - 2\alpha}_{O\hat{A}B} = 180^\circ$

$\Rightarrow O\hat{A}B = \alpha$

$\Rightarrow \ell = AB \Rightarrow \ell$ touches circle at 2 points: A, B

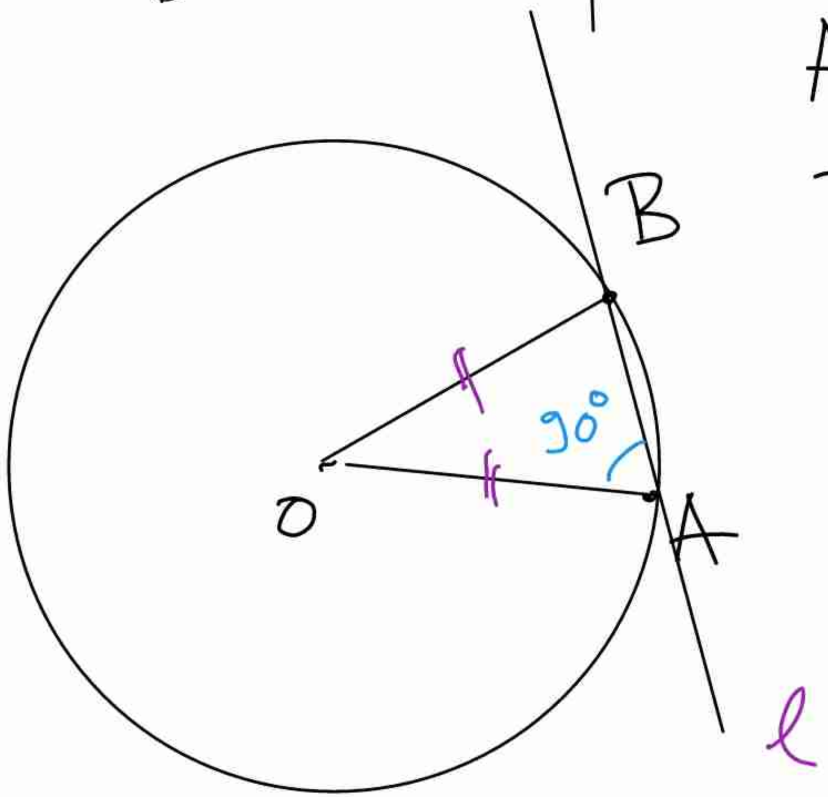
(contradiction) $\square$

$(\Leftarrow)$



Construct ℓ thru A such that $\ell \perp OA$

WTS: ℓ tangent to $\mathscr{C}$



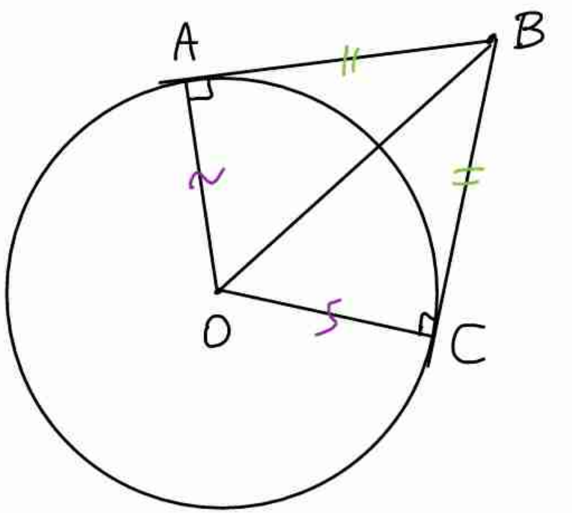
Assumme towards a contradiction that there exists another point B on both ℓ and $\mathscr{C}$

$\triangle OAB$ isosceles $\Rightarrow \widehat{OBA} = 90^\circ$

$\widehat{OAB} = 90^\circ$

$$\left.\begin{array}{l} \Rightarrow \widehat{OAB} + \widehat{OBA} = 180^\circ \\ \qquad \widehat{BOA} > 0 \end{array}\right\} \Rightarrow 180^\circ > 180^\circ \text{ contradiction}$$

$\Rightarrow \ell \cap \mathscr{C}$ has just one point, namely A. $\square$

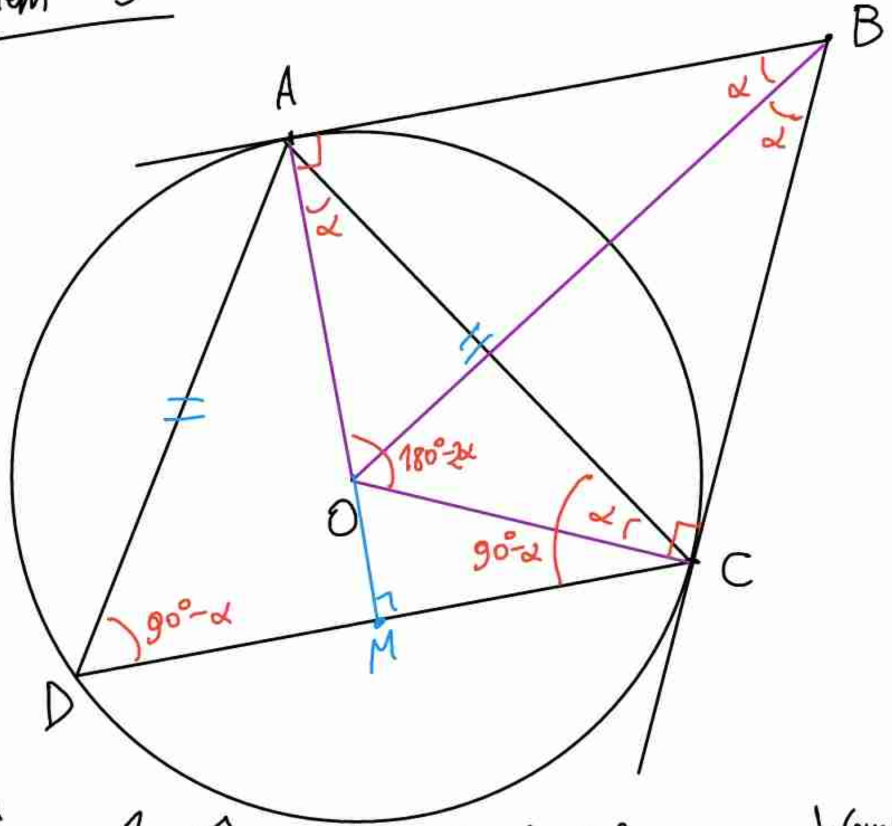


Suppose BA, BC tangent to circle;
then:

$$\triangle AOB \equiv \triangle COB \quad \text{(SS for right triangles)}$$

$$\Rightarrow AB = CB, \quad \widehat{AOB} = \widehat{COB}, \text{ etc.}$$

Problem 5



Know: $AB \parallel CD$,
BA, BC tangent to the circumcircle (ACD)

Want: $AC \overset{?}{=} AD$

$$\begin{cases} \triangle AOB \equiv \triangle COB \Rightarrow \widehat{ABO} = \widehat{CBO} =: \alpha \\ OABC \text{ cyclic} \Rightarrow \widehat{OAC} = \widehat{OCA} = \alpha \end{cases}$$

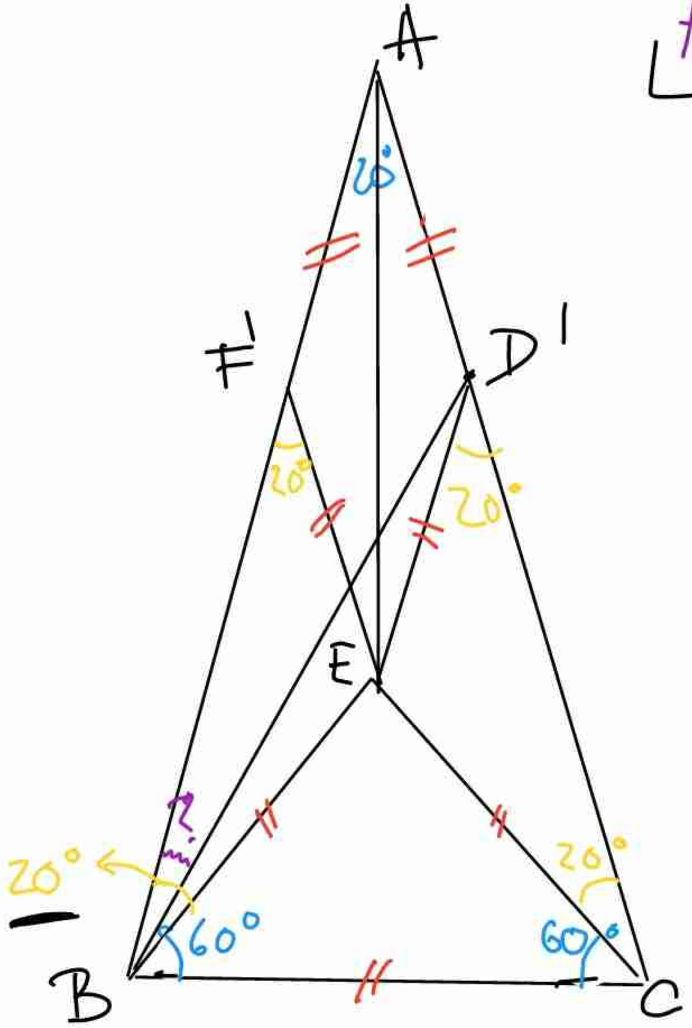
$$AB \parallel CD \Rightarrow \underset{\overset{\parallel}{90^\circ - \widehat{OAC}}}{\widehat{BAC}} = \underset{\overset{\parallel}{90^\circ - \alpha}}{\widehat{ACD}}$$

Want: $\widehat{ADC} \overset{?}{=} 90^\circ - \alpha$ (then $\triangle DAC$ would be isosc.)

But, $\widehat{ADC} = \frac{1}{2}\widehat{AOC} = \frac{1}{2}(180^\circ - 2\alpha) = 90^\circ - \alpha$.

Problem 7

$\widehat{ABD} = ?$



ABED' cyclic $\Rightarrow \widehat{ABD'} = \widehat{AED'} = \frac{180° - \widehat{AD'E}}{2} = \frac{\widehat{ED'C}}{2}$

ΔAED' iso with AD' = ED'

$= \frac{20°}{2} = \boxed{10°}$

Hint # 1

Construct ΔBCE equilateral

Take another point $D' \in AC$ such that $ED' = EC$

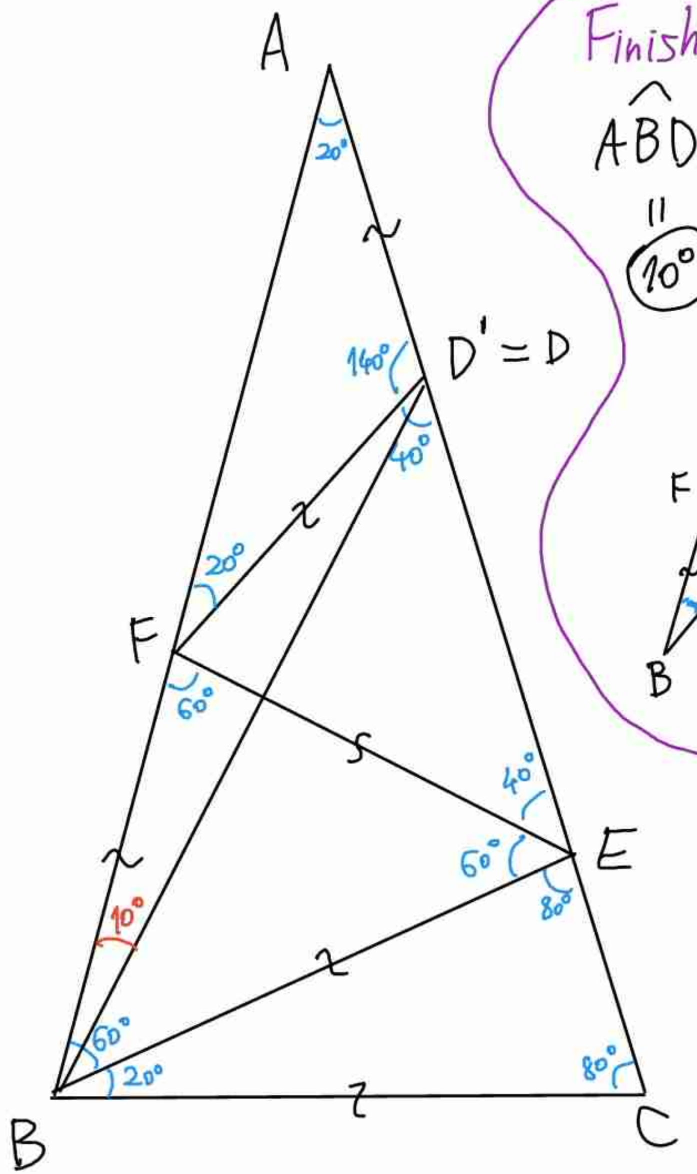
$\Rightarrow ED' \parallel AB$; similarly, $F' \in AB$ ...

$\Rightarrow AF'ED'$ parallelogram

$\Rightarrow AD' = BC \Rightarrow D'$ coincides with D

$\widehat{ABE} = 20°$, $\widehat{AD'E} = 180° - 20 \Rightarrow ABED'$ cyclic

Problem 7 (Soln. 2):



Hint #2: E on AC, ΔEBC isosc. at B

F on AB similarly

↳ such that $BF = BE$

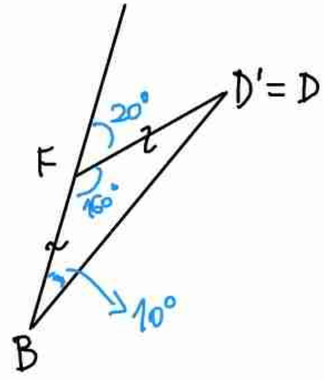
($\Rightarrow \Delta FBE$ equil., since $F\hat{B}E = 60°$)

Finish:

$A\hat{B}D = F\hat{B}D'$

$= 10°$

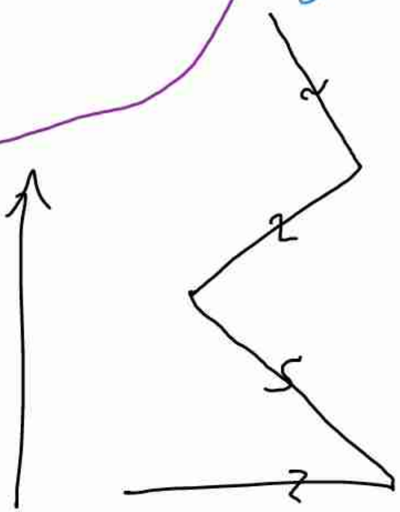
$\Delta BFD'$ isosc.



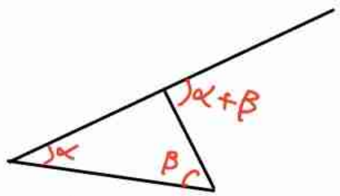
D' on AC, $FD' = FE$

Want to show $D' = D$

(meaning $AD' = AD$
$= BC$
$= FD'$)

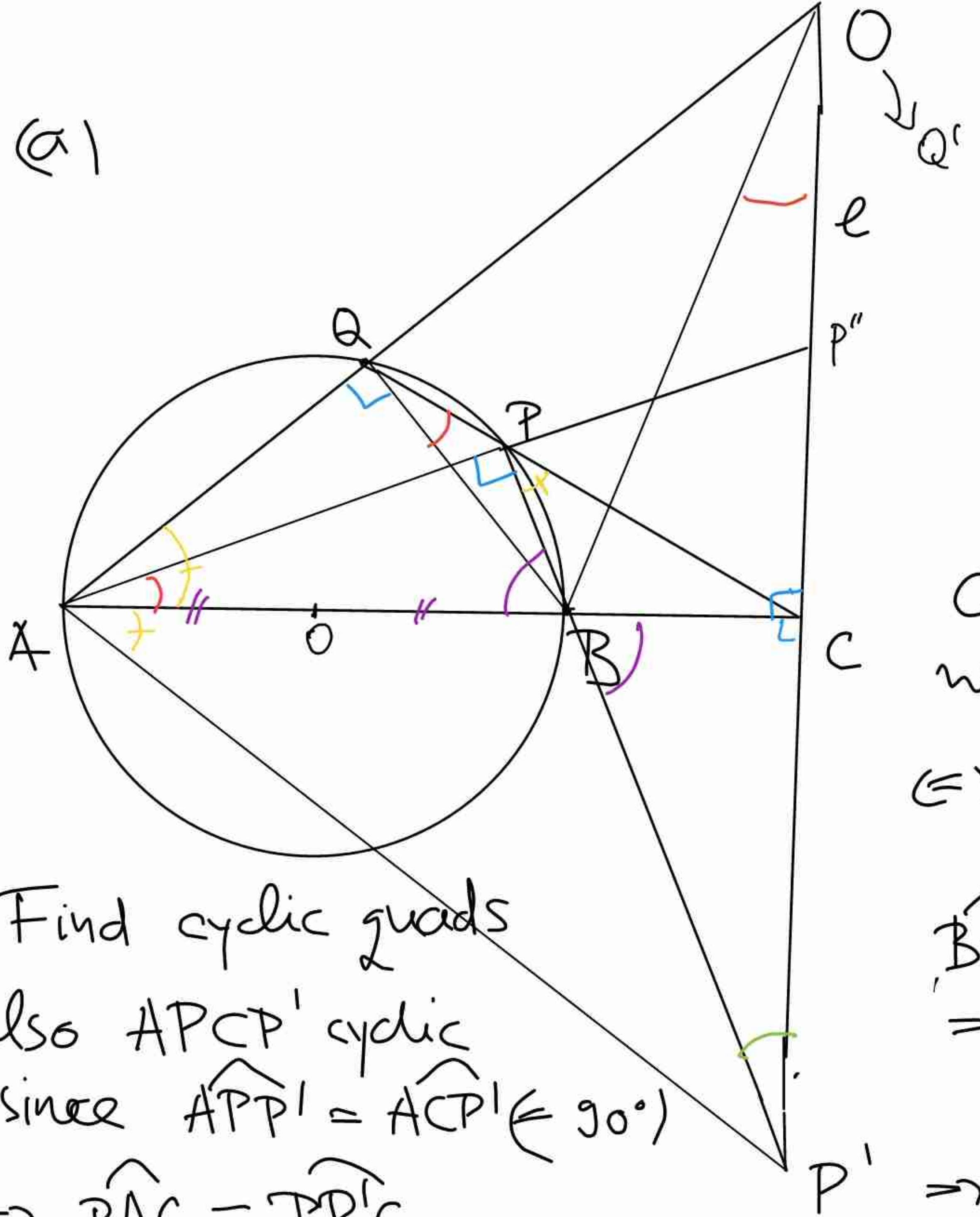


Angle chasing $\Rightarrow \Delta AD'F$ isosc. at $D' \Rightarrow AD' = BC$



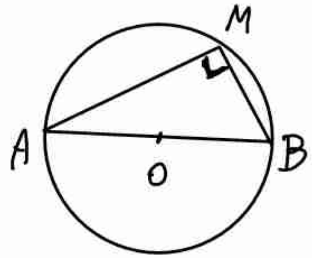
$\Rightarrow D' = D$.

8 (a)



$\boxed{P'C \overset{?}{=} Q'C}$ Show

(Know: AB diameter)



Another idea: show $\widehat{Q'AC} \overset{?}{=} \widehat{P'AC}$

Construct BQ';

WTS: $\underbrace{P'B \overset{?}{=} BQ'}$

$\Leftarrow \widehat{BP'C} = \widehat{BQ'C}$

$\widehat{BQQ'} = \widehat{BCQ'} = 90°$

$\left.\begin{array}{l} \Rightarrow QBCQ' \text{ cyclic} \\ ABPQ \text{ cyclic} \end{array}\right\} \Rightarrow$

$\Rightarrow \widehat{BQ'C} = \widehat{BQC} = \widehat{PAC}$

Find cyclic quads

also $APCP'$ cyclic

since $\widehat{APP'} = \widehat{ACP'} (= 90°)$

$\Rightarrow \widehat{PAC} = \widehat{PP'C}$