ORMC: ANGLE CHASING (AND SOME CONSTRUCTIONS)

OLYMPIAD GROUP 1, WEEK 8

[*Intro: Angles at the orthocenter.*]

Problem 1. Let $\triangle ABC$ be an acute scalene triangle. We let D be the projection of A on the line segment BC, and let E and F be the reflections of D with respect to AC and AB respectively. Find the angle between the lines EF and BC, in terms of the angles $\hat{A}, \hat{B}, \hat{C}$ of the initial triangle.

[*Interlude: Angles in circles.*]

Problem 2. Let $\mathscr{C}$ be a circle with center O, and let AB be a diameter. Let P be a point outside $\mathscr{C}$, and let X, Y be the intersection points of $\mathscr{C}$ with the line segments AP and BP respectively. If $\angle XPY = \angle XOY$, find their common value. (The answer should be a constant that does not depend on P)

[*Interlude: Cyclic quadrilaterals; angles of the orthic triangle.*]

Problem 3. Let $ABCD$ be a parallelogram with $AB > AD$ and $\hat{D} < 90°$, and let P be inside it such that $PA \perp AB$ and $PC \perp BC$. Show that $\angle PDC = \angle PBC$.

Problem 4. Let $ABCD$ be a square and P be a point inside it. Suppose that $\angle APD + \angle BPC = 180°$. Show that P lies on one of the square's diagonals (AC or BD).

[*Interlude: Tangents to circles.*]

Problem 5. Let $ABCD$ be a trapezoid with $AB \parallel CD$, and assume that AB and BC are both tangent to the circumcircle (ACD). Show that $AC = AD$.

Problem *6. Consider a convex quadrilateral $ABCD$ whose angles are

$$\hat{A} = 75°, \qquad \hat{B} = 45°, \qquad \hat{C} = 150°, \qquad \hat{D} = 90°,$$

and such that $BC = CD$. Show that $\angle BAC = 30°$. *Hint: reflect C across BD.*

Problem *7. Let $\triangle ABC$ be a triangle with $\hat{A} = 20°$ and $\hat{B} = \hat{C} = 80°$. Let D lie on the segment AC such that $AD = BC$. Compute the angle $\angle ABD$.

Idea 1: Construct E inside $\triangle ABC$ such that $\triangle BEC$ is equilateral; then construct D' on AC such that $ED' = EC$. Show that $D = D'$, and continue from here.

Idea 2: Construct E on AB such that $\triangle EBC$ is isosceles with $EB = BC$. Then pick F on AB such that $\triangle FEB$ is isosceles, and D' on BC such that $AD' = FD'$. Show that $D = D'$, and continue from here,

Problem 8. Let A, B, C be collinear points in this order. Suppose that points P, Q are on the circle of diameter AB such that C, P, Q are also collinear in this order. Let ℓ be the perpendicular through C to AC

(a) Take $\{P'\} = \ell \cap BP$, $\{Q'\} = \ell \cap AQ$. Show that C is the midpoint of $P'Q'$.

(b) Take $\{P''\} = \ell \cap AP$, $\{Q''\} = \ell \cap BQ$. Show that C is the midpoint of $P''Q''$.

(c) Let R and S be the reflections of P and Q with respect to AB. Show that Q', B, R are collinear, and so are P'', B, S, without using orthocenters.

In particular, this shows that the heights in triangles $\triangle AQ'Q''$ and $\triangle AP'P''$ are concurrent at B. One could have started from any of these triangles and constructed the other points, so this gives another proof that orthocenters exist (at least for acute triangles).