Finite Geometry

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1 Introduction

Definition 1.1 *Define a plane geometry to be a set P, thought of as a set of points, together with a set L of subsets of P, thought of as lines.*

One obvious example is the set $P = \mathbb{R}^2$, together with L the set of lines in the plane. Another is the real projective plane $\mathbb{RP}^2$, with its own set L of lines, any two of which intersect at exactly one point. However, in this worksheet, we will try to study plane geometries where P is finite, that still behave like the set of points and lines in a plane.

Problem 1 For an example of a finite plane geometry, let P be a set of 4 points, and let L be the set of all pairs of 2 points in P.

- How many lines are there?
- Can you draw a picture of this, representing each point in P with a dot, and each line in L with a line through the correct points?
- How many points are on a line?
- How many lines go through a given point?
- For a given line ℓ, how many other lines intersect it?
- As usual, we say that two lines are *parallel* if they do not intersect. In normal Euclidean geometry, for any line ℓ and any point p not on ℓ, there exists a unique line through p that is *parallel* to ℓ. Does this hold for this finite geometry?

Problem 2 Can you extend the plane geometry from the previous problem by adding three points and one line (and potentially extending existing lines) so that the following properties are satisfied?

- As in normal Euclidean geometry, any two distinct points determine a (unique) line.

- As in normal Euclidean geometry, there exists a set of four points, no three points of which are on the same line.
- Unlike in normal Euclidean geometry, every two distinct lines intersect at a unique point.

What other properties do you observe?

2 Affine Planes

Definition 2.1 *A* finite affine plane *is a plane geometry satisfying the following axioms:*

- *Every two distinct points determine a unique line.*
- *Any point not on a line ℓ is on precisely one line that does not intersect ℓ.*
- *There exist four points, no three of which are on the same line.*
- *There are only finitely many points.*

Problem 3 Let p be a prime. Let $P = \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, p-1\}$. Now let L be the set of lines of the form $ax + by \equiv c \mod p$, where as before, we require that a and b are not both 0.

Verify that this is a finite affine plane. We call this $A(2, p)$. Draw a picture of $A(2, 2)$ and $A(2, 3)$. Compare this to the plane geometry from Problem 1.

Theorem 2.1 *In any finite affine plane, the following hold, for some positive integer n (called the order of the finite affine plane):*

1. *If two lines are parallel to a third, they are parallel to each other.*
2. *Every line contains at least n points.*
3. *The total number of points in the plane is n^2 .*
4. *Each line intersects n^2 other lines.*
5. *Each line is parallel to n lines (including itself).*
6. *The total number of lines in the plane is $n^2 + n$.*

We will prove Theorem 2.1 over the course of several problems.

Problem 4 Check that Theorem 2.1 holds for $A(2, p)$ for any prime p.

Problem 5 Prove that all the lines in a finite affine plane must have the same number of points.

Problem 6 Prove that there cannot be seven distinct straight lines in a finite affine plane so that there are at least six points where three of them intersect, and at least four points where exactly two of them intersect.

Problem 7 The game Set consists of a deck of cards, each with 1, 2, or 3 symbols. The symbols on a card are all red, green, or purple. They are all ovals, diamonds, or squiggles. They are all filled in, shaded, or empty. To score in the game, you must find 3 cards such that, in each aspect (number, color, shape, and shading),either all three are the same, or all three are different. 3 such cards are called a "set." While there are 81 cards in a set deck (one for each possible choice of characteristics), can you find a collection of cards such that, if the cards are points and the "sets" are lines, you end up with an affine plane?

3 Projective Planes

Definition 3.1 *A finite projective plane is a plane geometry satisfying the following axioms:*

- *Every two distinct points determine a unique line.*
- *Every two distinct lines meet at a unique point.*
- *There exist four points, no three of which are on the same line.*
- *There are only finitely many points.*

Problem 8 Check that the plane geometry from Problem 2 is a projective plane.

Theorem 3.1 *In any finite projective plane, the following hold, for some positive integer n (called the order of the finite projective plane):*

- *All the points of the plane are on all the lines through any given point.*
- *Every line contains $n + 1$ points.*
- *The total number of points in the plane is $n^2 + n + 1$.*
- *Every point is on $n + 1$ lines.*
- *The total number of lines in the plane is $n^2 + n + 1$.*

We will prove Theorem 3.1 over the course of several problems.

Problem 9 A city bus network has the property that one can get from any bus stop to any other without transferring, for any pair of routes there is exactly one stop where one can transfer between them, and there are exactly three bus stops on each route.

How many bus routes are there in town?

Problem 10 Prove that in a projective plane, there are four lines, no three of which pass through the same point.

Problem 11 Prove that all the lines in a projective plane have the same number of points as follows:

1. Pick a line m, and say that it has k points on it. Explain why there must be at least two points not on that line.
2. Call the two points not on m P and Q; explain why there must be a line other than m that doesn't go through both P and a point on m (to do so, it will go through Q); call this line ℓ.
3. Explain why each of the lines that contain P and intersect m must intersect ℓ at a different point. How many points does this imply that ℓ has?
4. Reverse the above between m and ℓ to get the other inequality.
5. Explain why any line not passing through P must have exactly k points.
6. Explain why any line not passing through Q must have exactly k points.
7. Explain why PQ must have exactly k points.

Problem 12 In a particular city, there are 57 bus routes. One can get from any bus stop to any other, for any pair of routes there is exactly one stop where one can transfer between them, and every route has at least 3 stops.

How many stops are there on each bus route?

Problem 13 Prove that for any theorem about a projective plane, if you switch the words "line" and "point," you get another true statement. (Hint: prove that the switched version of each axiom holds. Then reason that since all theorems come from the axioms, however you would prove one theorem, you could use the switched axioms to prove the other.) Use this to prove the parts of Theorem 1 that you haven't already shown.

Problem 14 In the game Spot It!, two cards with some number of symbols on them are placed out, and players must determine what symbol is on both cards. So, for the game to work, each pair of cards must share exactly one symbol. So as to not be wasting space, each symbol needs to be on at least two cards. And, to prevent a too-easy round, each card needs to contain at least two symbols. And, to prevent the empty game, there must be some number of symbols. Lastly, each card must have the same number of symbols.

1. Draw a collection of cards that follows all the rules of Spot It! except for the last.

2. Give an example of a potential collection of Spot it! cards that follows all the rules of Spot It!. but if you replace cards and symbols with lines and points, you don't get a projective plane.

3. Give one additional condition on Spot It! cards that makes the rules of Spot It! equivalent to the axioms of a finite projective plane. Prove that equivalence.

4. How many cards are there in a Spot it! deck?

Problem 15 Let P and L define a finite projective plane, and let $\ell \in L$ be a line. Let $P' = P \setminus \ell$ (the set of all points in P not on ℓ), and let $L' = L \setminus \{\ell\}$ (the set of all lines in L other than ℓ).

- Show that P' and L' define a finite affine plane.
- Conclude that if n is the order of a finite projective plane, then it is also the order of a finite affine plane.

Problem 16 Let P and L determine a finite affine plane. Now let $\mathcal{E}$ be the set of equivalence classes of lines in L under the equivalence relation $\ell \sim m$ iff ℓ is parallel to m. Now let $P' = P \cup \mathcal{E}$. For each line $\ell \in L$, define a new line $\ell' \subset P'$ by $\ell' = \ell \cup \{E_\ell\}$, where $E_\ell \in \mathcal{E}$ is the equivalence class of ℓ. Now let $L' = \{\ell' : \ell \in L\} \cup \{\mathcal{E}\}$.

- Show that $\sim$ is actually an equivalence relation.
- Show that P' and L' define a finite projective plane.
- Conclude that n is the order of a finite affine plane if and only if it is the order of a finite projective plane.

4 Fields

The first several problems from this section were also in the challenge section from last week's worksheet, so skip any that you may have already done.

A *field* is a set $\mathbb{F}$ with addition, subtraction, multiplication, and inverse operations and special elements 0 and 1 such that the following properties hold:

Addition and subtraction properties:

- Associativity: For all $a, b, c \in \mathbb{F}$, $(a + b) + c = a + (b + c)$
- Commutativity: For all $a, b \in \mathbb{F}$, $a + b = b + a$
- Identity: For all $a \in \mathbb{F}$, $a + 0 = a$
- Inverse: For all $a, b \in \mathbb{F}$, $a + (b - a) = b$

Multiplication and inverse properties:

- Associativity: For all $a, b, c \in \mathbb{F}$, $(a * b) * c = a * (b * c)$
- Commutativity: For all $a, b \in \mathbb{F}$, $a * b = b * a$
- Identity: For all $a \in \mathbb{F}$, $a * 1 = a$
- Inverse: For all $a \in \mathbb{F}$, if $a \neq 0$, then $a * a^{-1} = 1$.

Distributivity: For all $a, b, c \in \mathbb{F}$, $a * (b + c) = a * b + a * c$.

Problem 17 Which of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields?

Problem 18 We will now define a field $\mathbb{Z}_p$ for each prime p. Let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$. It already comes with 0 and 1, and we can define $+, -, *, ^{-1}$ by addition, subtraction, multiplication and inverse mod p.

Verify that the axioms are satisfied.

Problem 19 For all positive $n \in \mathbb{N}$, we can define $+, -, *$ on $\{0, 1, \ldots, n-1\}$ by addition, subtraction, and multiplication mod n, like in the previous problem. Can we define an inverse operation that makes this set a field?

Problem 20 Show that if there exists a finite field with n elements, then there exists a finite affine plane and a finite projective plane of order n. The converse is an open problem!

Problem 21 Let $\mathbb{F} = \{0, 1, a, b\}$. Can you define $+, -*, ^{-1}$ such that $\mathbb{F}$ is a field?

Problem 22 Let $\mathbb{F}$ be a field. If there exists a positive integer n such that $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$, then we call the smallest such number the *characteristic* of $\mathbb{F}$. Otherwise, we say that the characteristic of $\mathbb{F}$ is 0.

- Find the characteristics of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{Z}_p$ for all primes p.
- Show that the characteristic of $\mathbb{F}$ is either 0 or a prime.

Problem 23 Show that there is no finite field with 6 elements.

(Hint: Assume for contradiction that such a field exists. If it has characteristic 2, name its elements $\{0, 1, a, a+1, b, b+1\}$. What is $a + a$? What is $b + b$? Then what can $a + b$ be? If it has characteristic 3, name its elements $\{0, 1, 2, a, a+1, a+2\}$. What is $a + a$?)