

ORMC: ANGLE CHASING (AND SOME CONSTRUCTIONS)

OLYMPIAD GROUP 1, WEEK 8

[Intro: Angles at the orthocenter.]

Problem 1. Let $\triangle ABC$ be a scalene triangle. We let D be the projection of A on the line segment BC , and let E and F be the reflections of D with respect to AC and AB respectively. Find the angle between the lines EF and BC , in terms of the angles $\hat{A}, \hat{B}, \hat{C}$ of the initial triangle.

[Interlude: Angles in circles.]

Problem 2. Let \mathcal{C} be a circle with center O , and let AB be a diameter. Let P be a point outside \mathcal{C} , and let X, Y be the intersection points of \mathcal{C} with the line segments AP and BP respectively. If $\angle XPY = \angle XOY$, find their common value. (The answer should be a constant that does not depend on P)

[Interlude: Cyclic quadrilaterals; angles of the orthic triangle.]

Problem 3. Let $ABCD$ be a parallelogram with $AB > AD$ and $\hat{D} < 90^\circ$, and let P be inside it such that $PA \perp AB$ and $PC \perp BC$. Show that $\angle PDC = \angle PBC$.

Problem 4. Let $ABCD$ be a square and P be a point inside it. Suppose that $\angle APD + \angle BPC = 180^\circ$. Show that P lies on one of the squares diagonals (AC or BD).

[Interlude: Tangents to circles.]

Problem 5. Let $ABCD$ be a trapezoid with $AB \parallel CD$, and assume that AB and BC are both tangent to the circumcircle (ACD). Show that $AC = AD$.

Problem *6. Consider a convex quadrilateral $ABCD$ whose angles are

$$\hat{A} = 75^\circ, \quad \hat{B} = 45^\circ, \quad \hat{C} = 150^\circ, \quad \hat{D} = 90^\circ,$$

and such that $BC = CD$. Show that $\angle BAC = 30^\circ$. *Hint: reflect C across BD .*

Problem *7. Let $\triangle ABC$ be a triangle with $\hat{A} = 20^\circ$ and $\hat{B} = \hat{C} = 80^\circ$. Let D lie on the segment AC such that $AD = BC$. Compute the angle $\angle ABD$.

Hint: Construct E on AC such that $\triangle EBC$ is isosceles. Then pick F on AB similarly.